有限元方法

Finite Element Methods

Chapter 7: Galerkin Approach for Time-Dependent Problems

主讲人: 李琦

liqihao@chd.edu.cn

School of Science, Chang'an University

The Standard Galerkin Method for the Heat Equation

We consider the model heat equation:

Heat equation

$$(\mathcal{P}) \qquad \begin{cases} u_t - \Delta u = f, & \text{in } \Omega, \ t > 0, \\ u = 0, & \text{on } \partial\Omega, \\ u(\cdot, 0) = v. \end{cases}$$

Goals of the chapter:

- Introduce the standard Galerkin finite element method.
- Study spatially semidiscrete approximations.
- Construct fully discrete schemes in time.
- Derive optimal error estimates.

Strategy: PDE $\Longrightarrow$ weak formulation $\Longrightarrow$ finite-dimensional problem

Main time discretizations:

- Backward Euler method; Crank–Nicolson method;
- Second-order backward difference method (BDF2).

Sobolev Spaces and Norms

For $\Omega \subset \mathbb{R}^d$,

$$L_2(\Omega) = \{v : \Omega \to \mathbb{R} \mid \|v\| < \infty\},$$

with norm

$$\|v\| = \left(\int_\Omega v^2 \, dx \right)^{1/2}.$$

For integer $r \geq 1$,

$$H^r(\Omega) = W_2^r(\Omega),$$

with norm

$$\|v\|_r = \left(\sum_{|\alpha| \leq r} \|D^\alpha v\|^2 \right)^{1/2}.$$

Here

$$D^\alpha = \left(\frac{\partial}{\partial x_1} \right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_d} \right)^{\alpha_d}, \qquad |\alpha| = \sum_{j=1}^{d} \alpha_j.$$

For functions in $H_0^1(\Omega)$,

$$c\|v\|_1 \leq \|\nabla v\| \leq \|v\|_1.$$

This follows from Friedrichs' inequality. $\|\nabla v\|$ and $\|v\|_1$ are equivalent norms.

Weak Formulation of Poisson's Equation

Consider

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

Multiply by a test function $\varphi \in H_0^1$ and integrate over Ω:

$$-\int_\Omega \Delta u \, \varphi \, dx = \int_\Omega f \varphi \, dx.$$

Using Green's formula:

$$(\nabla u, \nabla \varphi) = (f, \varphi), \qquad \forall \varphi \in H_0^1.$$

Inner products:

$$(v, w) = \int_\Omega vw \, dx, \quad (\nabla v, \nabla w) = \int_\Omega \sum_{j=1}^{d} \frac{\partial v}{\partial x_j} \frac{\partial w}{\partial x_j} \, dx.$$

A function $u \in H_0^1$ satisfying $(\nabla u, \nabla \varphi) = (f, \varphi)$ is called the variational solution.

Elliptic Regularity

It is an easy consequence of the Riesz representation theorem that a unique such solution exists if $f \in H^{-1}$, the dual space of H_0^1.

If $\partial\Omega$ is smooth, then the solution gains two derivatives:

$$\|u\|_{m+2} \leq C\|\Delta u\|_m = C\|f\|_m, \qquad m \geq -1.$$

In particular:

$$f \in H^m \implies u \in H^{m+2}.$$

If $f \in C^\infty$, then $u \in C^\infty$.

Important consequence:

- Smooth right-hand side $\Rightarrow$ smooth solution.
- Regularity plays a key role in error estimates.

For convex polygonal domains: $\|u\|_2 \leq C\|f\|$.

For nonconvex polygonal domains, this may fail.

Triangulation and Finite Element Space

Let thus Ω be a convex domain in the plane with smooth boundary $\partial\Omega$, and let $\mathcal{T}_h$ be a triangulation of Ω. Define

$$h = \max_{\tau \in \mathcal{T}_h} \operatorname{diam}(\tau).$$

Assumptions:

- Angles are bounded below.
- Sometimes quasiuniformity is assumed: $|\tau| \geq ch^2$.

Finite element space:

$$S_h = \left\{ \chi \in C(\bar{\Omega}) : \chi|_\tau \text{ linear}, \ \chi = 0 \text{ on } \partial\Omega \right\}.$$

Let $P_1, \dots, P_{N_h}$ be interior vertices. Define basis functions Φ_j: $\Phi_j(P_i) = \delta_{ij}$. Then

$$\chi(x) = \sum_{j=1}^{N_h} \alpha_j \Phi_j(x).$$

The coefficients satisfy $\alpha_j = \chi(P_j)$.

Interpolation Operator

For a smooth function v, define the interpolant

$$I_h v(x) = \sum_{j=1}^{N_h} v(P_j)\Phi_j(x).$$

Then $I_h v(P_j) = v(P_j)$.

Interpolation error estimates:

$$\|I_h v - v\| \le Ch^2 \|v\|_2, \quad \|\nabla(I_h v - v)\| \le Ch\|v\|_2.$$

More generally, when $v \in H^s(\Omega) \cap H_0^1$,

$$\inf_{\chi \in S_h} \{\|v-\chi\| + h\|\nabla(v-\chi)\|\} \le Ch^s \|v\|_s,$$

for $1 \le s \le r$.

The number r is referred to as the order of accuracy of the family $\{S_h\}$. An interpolation operator $I_h : H^r \cap H_0^1 \to S_h$ such that

$$\|I_h v - v\| + h\|\nabla(I_h v - v)\| \le Ch^s \|v\|_s, \quad \text{for } 1 \le s \le r.$$

Inverse Inequality

For quasiuniform triangulations, finite element functions satisfy the inverse estimate:

$$\|\nabla\chi\| \leq Ch^{-1}\|\chi\|, \qquad \forall\chi \in S_h.$$

Key idea of proof:

- Transform each element to a reference triangle.
- Use equivalence of norms on finite-dimensional spaces.
- Sum over all elements.

Interpretation:

Higher derivatives $\sim h^{-1} \times$ lower derivatives.

This estimate is crucial in:

- stability analysis,
- fully discrete schemes,
- maximum norm estimates.

Galerkin Approximation for Poisson's Equation

Find $u_h \in S_h$ such that

$$(\nabla u_h, \nabla \chi) = (f, \chi), \qquad \forall \chi \in S_h.$$

Galerkin orthogonality:

$$(\nabla(u_h - u), \nabla \chi) = 0, \qquad \forall \chi \in S_h.$$

Using basis functions:

$$u_h(x) = \sum_{j=1}^{N_h} \alpha_j \Phi_j(x).$$

Then

$$\sum_{j=1}^{N_h} \alpha_j (\nabla \Phi_j, \nabla \Phi_k) = (f, \Phi_k), \qquad k = 1, \dots, N_h.$$

Matrix form:

$$\mathcal{A}\alpha = \widetilde{f},$$

where $a_{jk} = (\nabla \Phi_j, \nabla \Phi_k)$.

$\mathcal{A}$ is symmetric positive definite.

Error Estimates for the Elliptic Problem

Theorem (1.1)

Assume that

$$\inf_{\chi \in S_h} \{\|u - \chi\| + h\|\nabla(u - \chi)\|\} \leq Ch^s \|u\|_s.$$

holds. Then, for $1 \leq s \leq r$*,*

$$\|u_h - u\| \leq Ch^s \|u\|_s \quad \textit{and} \quad \|\nabla u_h - \nabla u\| \leq Ch^{s-1} \|u\|_s.$$

- Energy norm estimate:

$$\|\nabla(u_h - u)\| \leq \inf_{\chi \in S_h} \|\nabla(\chi - u)\|.$$

- L_2-estimate uses the duality argument: Let $-\Delta\psi = \varphi$. Then

$$(u_h - u, \varphi) = (\nabla(u_h - u), \nabla(\psi - \psi_h)).$$

Apply:

- elliptic regularity,
- interpolation estimate,
- Galerkin orthogonality.

Weak Solution of the Heat Equation

We consider the initial-boundary value problem

$$u_t - \Delta u = f \quad \text{in } \Omega,\ t > 0, \qquad u = 0 \quad \text{on } \partial\Omega,\ t > 0, \qquad u(\cdot, 0) = v.$$

We say that a function $u = u(x,t)$ is a weak solution on $[0, \bar{t}]$ if

$$(u_t, \varphi) + (\nabla u, \nabla \varphi) = (f, \varphi), \qquad \forall \varphi \in H_0^1,$$

holds for $t > 0$, with

$$u \in L_2(0, \bar{t}; H_0^1), \qquad u_t \in L_2(0, \bar{t}; H^{-1}),$$

and with initial value

$$u(\cdot, 0) = v.$$

Here

$$(v, w) = \int_\Omega vw \, dx, \qquad (\nabla v, \nabla w) = \int_\Omega \sum_{j=1}^{d} \frac{\partial v}{\partial x_j} \frac{\partial w}{\partial x_j} \, dx.$$

Parabolic Regularity Estimate

Since the boundary $\partial\Omega$ is smooth, the weak solution is smooth provided the data are smooth.

This smoothness is expressed by the parabolic regularity estimate

$$\sum_{j=0}^{m+1} \int_0^{\bar{t}} \|u^{(j)}\|_{2(m-j)+2}^2 \, dt \le C \left(\|v\|_{2m+1}^2 + \sum_{j=0}^{m} \int_0^{\bar{t}} \|f^{(j)}\|_{2(m-j)}^2 \, dt \right), \qquad m \ge 0,$$

where

$$u^{(j)} = \left(\frac{\partial}{\partial t} \right)^j u, \qquad C = C_{m,\bar{t}}.$$

Thus, sufficiently smooth initial data v and source term f yield higher regularity of both time derivatives and spatial derivatives of u.

In particular, the solution gains spatial smoothness in a way consistent with the elliptic regularity of the heat operator.

Compatibility Conditions at $t = 0$

The regularity estimate above is valid only when the data satisfy suitable compatibility conditions at $t = 0$.

These conditions express that the boundary condition and the initial condition are consistent with each other.

First compatibility condition ($m = 0$): since

$$u(t) = 0 \quad \text{on } \partial\Omega, \qquad t > 0,$$

we must also have $u(0) = v = 0 \quad \text{on } \partial\Omega.$

Second compatibility condition ($m = 1$): Since the boundary condition holds for all $t > 0$, differentiation in time implies $u_t = 0, \text{ on } \partial\Omega$. Evaluating the PDE at $t = 0$,

$$u_t(0) = \Delta v + f(0),$$

hence smoothness requires

$$\Delta v + f(0) = 0 \quad \text{on } \partial\Omega.$$

Similarly, for higher time derivatives $u^{(m)}(0)$, further compatibility conditions are needed for $m \geq 2$.

Remarks

The above conditions are standard in the regularity theory for parabolic equations.

They ensure that:

- the prescribed initial value agrees with the boundary condition at $t = 0$,
- higher time derivatives are also consistent with the boundary data,
- the solution enjoys the smoothness needed in later finite element error analysis.

Hence, under smoothness of $\partial\Omega$, v, and f, together with the required compatibility conditions, the weak solution becomes sufficiently regular for the estimates used in the semidiscrete and fully discrete analyses.

Semidiscrete Galerkin Method

Spatially semidiscrete problem

Find $u_h(t) \in S_h$ such that

$$(\mathcal{W}_h) \qquad (u_{h,t}, \chi) + (\nabla u_h, \nabla \chi) = (f, \chi),$$

for all $\chi \in S_h$, $t > 0$, with $u_h(0) = v_h$.

Expand:

$$u_h(x,t) = \sum_{j=1}^{N_h} \alpha_j(t)\Phi_j(x).$$

Then

$$\sum_{j=1}^{N_h} \alpha_j'(t)(\Phi_j, \Phi_k) + \sum_{j=1}^{N_h} \alpha_j(t)(\nabla\Phi_j, \nabla\Phi_k) = (f, \Phi_k).$$

Matrix form:

$$\mathcal{B}\alpha'(t) + \mathcal{A}\alpha(t) = \widetilde{f}(t).$$

Mass and stiffness matrix:

$$b_{jk} = (\Phi_j, \Phi_k), \qquad a_{jk} = (\nabla\Phi_j, \nabla\Phi_k).$$

Ritz Projection

Define the Ritz projection $R_h : H_0^1 \to S_h$ by

$$(\nabla R_h v, \nabla \chi) = (\nabla v, \nabla \chi), \qquad \forall \chi \in S_h.$$

Interpretation:

- $R_h v$ is the finite element approximation of v.
- Orthogonal projection in the energy inner product.

Error splitting:

$$u_h - u = \theta + \rho,$$

where $\theta = u_h - R_h u$, $\rho = R_h u - u$.

Properties:

$$\|\nabla R_h v\| \leq \|\nabla v\|,$$

and

$$\|R_h v - v\| + h\|\nabla(R_h v - v)\| \leq C h^s \|v\|_s.$$

Semidiscrete Error Estimate

Theorem (1.2)

Let u_h and u be the solutions of $(\mathcal{W}_h)$ and $(\mathcal{P})$, and assume $v = 0$ on $\partial\Omega$. Then

$$\|u_h(t) - u(t)\| \leq \|v_h - v\| + Ch^r \left(\|v\|_r + \int_0^t \|u_t\|_r \, ds \right), \quad \text{for } t \geq 0.$$

Main idea:

$$u_h - u = (u_h - R_h u) + (R_h u - u) =: \theta + \rho.$$

The elliptic part:

$$\|\rho(t)\| \leq Ch^r \|u(t)\|_r.$$

For θ:

$$(\theta_t, \chi) + (\nabla\theta, \nabla\chi) = -(\rho_t, \chi).$$

Choose $\chi = \theta$. Then

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2 + \|\nabla\theta\|^2 = -(\rho_t, \theta).$$

After integration:

$$\|\theta(t)\| \leq \|\theta(0)\| + \int_0^t \|\rho_t\| \, ds.$$

Gradient Error Estimate

Theorem (1.3)

Let u_h and u be the solutions of $(\mathcal{W}_h)$ and $(\mathcal{P})$, and assume $v = 0$ on $\partial\Omega$. Then

$$\|\nabla u_h(t) - \nabla u(t)\| \le C\|\nabla(v_h - v)\| + Ch^{r-1}\left(\|v\|_r + \|u(t)\|_r + \left(\int_0^t \|u_t\|_{r-1}^2\, ds\right)^{1/2}\right).$$

Key step:

Choose $\chi = \theta_t$ in

$$(\theta_t, \chi) + (\nabla\theta, \nabla\chi) = -(\rho_t, \chi).$$

Then

$$\|\theta_t\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla\theta\|^2 = -(\rho_t, \theta_t).$$

Hence

$$\frac{d}{dt}\|\nabla\theta\|^2 \le \|\rho_t\|^2.$$

Integrating in time yields the estimate.

Superconvergence

Assume $v_h = R_h v$. Then $\theta(0) = 0$.

Using the previous estimate:

$$\|\nabla\theta(t)\| \leq C \left(\int_0^t \|\rho_t\|^2 \, ds \right)^{1/2}.$$

Since

$$\|\rho_t\| = \|R_h u_t - u_t\| \leq Ch^r \|u_t\|_r,$$

we obtain

$$\|\nabla\theta(t)\| \leq Ch^r \left(\int_0^t \|u_t\|_r^2 \, ds \right)^{1/2}.$$

Observation: ∇u_h approximates $\nabla R_h u$ with order $O(h^r)$, while ∇u_h approximates ∇u only with order $O(h^{r-1})$.

This phenomenon is called:

Superconvergence

Maximum Norm Error Estimates

For piecewise linear elements:

$$\|I_h v - v\|_{L_\infty} \le Ch^2 \|v\|_{W^2_\infty}.$$

Theorem (1.4)

Let u_h *and* u *be the solutions of* $(\mathcal{W}_h)$ *and* $(\mathcal{P})$*, and assume* $v = 0$ *on* $\partial\Omega$*. Then*

$$\|u_h - u\|_{L_\infty} \le Ch^2 \ell_h \|u\|_{W^2_\infty},$$

where $\ell_h = \max(1, \log(1/h))$.

Ritz projection stability:

$$\|R_h v\|_{L_\infty} \le C\ell_h \|v\|_{L_\infty}.$$

Parabolic estimate:

$$\|u_h(t) - u(t)\|_{L_\infty} \le C(t, u) h^2 \ell_h.$$

Important fact:

- The logarithmic factor ℓ_h cannot be removed.
- Nevertheless, it grows very slowly.

Backward Euler Galerkin Method

We introduce a time step k and the time levels $t_n = nk$ $(n \geq 0)$, and denote by $U^n = U_h^n \in S_h$ the approximation of $u(t_n)$.

Backward difference:

$$\bar{\partial} U^n = \frac{U^n - U^{n-1}}{k}.$$

Fully Discrete Backward Euler

Find $U^n \in S_h$ such that

$$(\mathcal{BE}) \qquad \left(\bar{\partial} U^n, \chi\right) + (\nabla U^n, \nabla \chi) = (f(t_n), \chi)\,, \qquad U^0 = v_h,$$

for all $\chi \in S_h$, $n \geq 1$.

Equivalent formulation:

$$(U^n, \chi) + k(\nabla U^n, \nabla \chi) = (U^{n-1} + kf(t_n), \chi).$$

Matrix form:

$$(\mathcal{B} + k\mathcal{A})\alpha^n = \mathcal{B}\alpha^{n-1} + k\widetilde{f}(t_n).$$

The matrix $\mathcal{B} + k\mathcal{A}$ is symmetric positive definite.

Error Estimate for Backward Euler

Theorem (1.5)

With U^n and u the solutions of $(\mathcal{BE})$ and $(\mathcal{P})$, respectively, we have, if $\|v_h - v\| \le Ch^r \|v\|_r$ and $v = 0$ on $\partial\Omega$,

$$\|U^n - u(t_n)\| \le Ch^r \left(\|v\|_r + \int_0^{t_n} \|u_t\|_r \, ds \right) + k \int_0^{t_n} \|u_{tt}\| \, ds, \qquad \text{for } n \ge 0.$$

Error decomposition: $U^n - u(t_n) = \theta^n + \rho^n.$

Discrete error equation:

$$(\bar{\partial}\theta^n, \chi) + (\nabla\theta^n, \nabla\chi) = -(\omega^n, \chi).$$

Consistency error:

$$\omega^n = (R_h - I)\bar{\partial}u(t_n) + (\bar{\partial}u(t_n) - u_t(t_n)).$$

Main result:

$$\boxed{\|U^n - u(t_n)\| = O(h^r + k)}$$

Backward Euler: • first order in time, • unconditionally stable.

Crank–Nicolson Galerkin Method

Define $\widehat{U}^n = \frac{1}{2}\left(U^n + U^{n-1}\right)$.

Fully Discrete Crank–Nicolson

Find $U^n \in S_h$ such that

$$(\mathcal{CN}) \qquad (\bar{\partial}U^n, \chi) + \left(\nabla\widehat{U}^n, \nabla\chi\right) = \left(f(t_{n-\frac{1}{2}}), \chi\right), \qquad U^0 = v_h,$$

for all $\chi \in S_h$, $n \geq 1$.

Matrix form: $\left(\mathcal{B} + \frac{1}{2}k\mathcal{A}\right)\alpha^n = \left(\mathcal{B} - \frac{1}{2}k\mathcal{A}\right)\alpha^{n-1} + k\widetilde{f}(t_{n-\frac{1}{2}})$.

Theorem (1.6)

Let U^n and u be the solutions of $(\mathcal{CN})$ and $(\mathcal{P})$, respectively, and let $\|v_h - v\| \leq Ch^r\|v\|_r$ and $v = 0$ on $\partial\Omega$. Then, for $n \geq 0$,

$$\|U^n - u(t_n)\| \leq Ch^r\left(\|v\|_r + \int_0^{t_n}\|u_t\|_r\,ds\right) + Ck^2\int_0^{t_n}\left(\|u_{ttt}\| + \|\Delta u_{tt}\|\right)ds.$$

Main result: $\boxed{\|U^n - u(t_n)\| = O(h^r + k^2)}$.

Second Order Backward Difference Method

We define the second order backward difference operator:

$$\bar{D}U^n = \frac{\frac{3}{2}U^n - 2U^{n-1} + \frac{1}{2}U^{n-2}}{k} u_t(t_n) + O(k^2).$$

Fully Discrete BDF2 Galerkin Method

Find $U^n \in S_h$ such that

$$(\mathcal{BDF}2) \qquad \left(\bar{D}U^n, \chi\right) + (\nabla U^n, \nabla\chi) = (f(t_n), \chi)\,, \qquad n \geq 2,$$

for all $\chi \in S_h$, with $U^0 = v_h$.

The first step U^1 is computed by one backward Euler step:

$$\left(\bar{\partial}U^1, \chi\right) + \left(\nabla U^1, \nabla\chi\right) = (f(t_1), \chi)\,.$$

Matrix form:

$$\left(\frac{3}{2}\mathcal{B} + k\mathcal{A}\right)\alpha^n = 2\mathcal{B}\alpha^{n-1} - \frac{1}{2}\mathcal{B}\alpha^{n-2} + k\widetilde{f}(t_n), \qquad n \geq 2.$$

Theorem (1.7)

Let U^n and u be the solutions of ($\mathcal{BDF}2$) and ($\mathcal{P}$), with $U^0 = v_h$ and U^1 defined as above. If $\|v_h - v\| \le Ch^r\|v\|_r$ and $v = 0$ on $\partial\Omega$, then

$$\|U^n - u(t_n)\| \le Ch^r\left(\|v\|_r + \int_0^{t_n}\|u_t\|_r\,ds\right) + Ck\int_0^k\|u_{tt}\|\,ds + Ck^2\int_0^{t_n}\|u_{ttt}\|\,ds, \qquad n \ge 0.$$

Main result:

$$\boxed{\|U^n - u(t_n)\| = O(h^r + k^2)}.$$

Variable Time Step Backward Euler Method

Let $0 = t_0 < t_1 < \cdots < t_n < \cdots ,\qquad k_n = t_n - t_{n-1}$. Define $\bar{\partial}_n U^n = \frac{U^n - U^{n-1}}{k_n}$.

Fully Discrete Variable-Step Backward Euler

Find $U^n \in S_h$ such that

$$(\mathcal{VBE}) \qquad (\bar{\partial}_n U^n, \chi) + (\nabla U^n, \nabla \chi) = (f(t_n), \chi), \qquad n \geq 1,$$

for all $\chi \in S_h$, with $U^0 = v_h$.

Theorem (1.8)

Let U^n and u be the solutions of ($\mathcal{VBE}$) and ($\mathcal{P}$), with $U^0 = v_h$ such that $\|v_h - v\| \leq Ch^r \|v\|_r$ and $v = 0$ on $\partial\Omega$. Then, for $n \geq 0$,

$$\|U^n - u(t_n)\| \leq Ch^r \left(\|v\|_r + \int_0^{t_n} \|u_t\|_r \, ds \right) + \sum_{j=1}^{n} k_j \int_{t_{j-1}}^{t_j} \|u_{tt}\| \, ds.$$

Advantages:

- adaptive time stepping, efficient for multiscale problems,
- larger steps when the solution varies slowly.