有限元方法

Finite Element Methods

Chapter 6: Mixed Finite Element Methods

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Outline

Definition

Reflexive A Banach space X is called **reflexive** if the canonical embedding

$$J : X \to X^{**}$$

is surjective, where X^* is the dual space of X and X^{**} is the bidual space. The canonical embedding J is defined by

$$(Jx)(f) = f(x), \qquad \forall x \in X,\ \forall f \in X^*.$$

Equivalently, a Banach space X is reflexive if

$$X \cong X^{**}.$$

$$\begin{tikzcd}
X \arrow[r, "f(x)"] \arrow[rr, bend left, "Jx(f) = f(x)"] & X^* \arrow[r] & X^{**} \\
\text{Banach space} & \text{dual space} & \text{bidual space}
\end{tikzcd}$$

Variational Problems with Constraints

Motivation

We consider variational problems with constraints, which arise in many applications:

- **Incompressible flow**: $\nabla \cdot u = 0$,
- **Weak enforcement of constraints**, e.g., boundary conditions.

Abstract Setting

Let V and M be reflexive Banach spaces.

- $a : V \times V \to \mathbb{R}$: symmetric and coercive,
- $F : V \to \mathbb{R}$: linear functional.

The unconstrained problem:

$$a(u, v) = F(v), \quad \forall v \in V,$$

is equivalent to minimizing the energy functional:

$$J(v) = \frac{1}{2} a(v, v) - F(v).$$

Lagrange Multiplier Formulation

Adding Constraints

We impose the constraint:

$$b(u, \mu) = 0, \quad \forall \mu \in M,$$

where

$$b : V \times M \to \mathbb{R}.$$

Example: incompressibility condition

$$b(u, \mu) = (\nabla \cdot u, \mu).$$

Lagrangian Functional

Introduce the Lagrangian:

$$L(u, \lambda) = J(u) + b(u, \lambda).$$

Saddle Point Problem

$$\inf_{v \in V} \sup_{\mu \in M} L(v, \mu).$$

First-Order Optimality Conditions

Stationarity Conditions

Taking variations with respect to v and μ, we obtain:

$$\begin{cases} a(u,v) + b(v,\lambda) = F(v), & \forall v \in V, \\ b(u,\mu) = 0, & \forall \mu \in M. \end{cases}$$

Remarks

- This is a saddle point problem
- λ is the Lagrange multiplier
- Existence of λ requires conditions on b

Next Step

These conditions will be characterized by the inf-sup condition.

Gâteaux Derivative: Definition and Examples

Definition (Gâteaux derivative). Let X be a Banach space and $J : X \to \mathbb{R}$. The Gâteaux derivative of J at $u \in X$ in the direction $v \in X$ is defined by

$$\delta J(u)[v] := \lim_{\epsilon \to 0} \frac{J(u + \epsilon v) - J(u)}{\epsilon},$$

provided the limit exists.

Example 1 (Quadratic energy). Let

$$J(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx.$$

Then $\delta J(u)[v] = \int_\Omega \nabla u \cdot \nabla v \, dx$.

Example 2 (Nonlinear potential). Let

$$J(u) = \int_\Omega F(u) \, dx,$$

where F is smooth. Then $\delta J(u)[v] = \int_\Omega F'(u) \, v \, dx$.

Remark. If $\delta J(u)[v] = 0$ for all $v \in X$, then u is a **critical point** of J.

Abstract Saddle Point Problem

Let V and M be reflexive Banach spaces.

We consider bilinear forms:

$$a : V \times V \to \mathbb{R}, \qquad b : V \times M \to \mathbb{R},$$

assumed continuous (not necessarily symmetric).

Given $f \in V^*$, $g \in M^*$, we seek $(u, \lambda) \in V \times M$ such that:

$$(\mathcal{S}) \qquad \begin{cases} a(u,v) + b(v,\lambda) = \langle f, v \rangle, & \forall v \in V, \\ b(u,\mu) = \langle g, \mu \rangle, & \forall \mu \in M. \end{cases}$$

This is called an **abstract saddle point problem**.

Operator Formulation

Define operators induced by bilinear forms:

$$\begin{aligned} A : V \to V^*, &\quad \langle Au, v\rangle = a(u,v), \\ B : V \to M^*, &\quad \langle Bu, \mu\rangle = b(u,\mu), \\ B^* : M \to V^*, &\quad \langle B^*\lambda, v\rangle = b(v,\lambda). \end{aligned}$$

Then the system becomes:

$$\begin{cases} Au + B^*\lambda = f \quad \text{in } V^*, \\ Bu = g \quad \text{in } M^*. \end{cases}$$

Block structure:

$$\begin{bmatrix} A & B^* \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ \lambda \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}.$$

First Observation: If B is invertible

If $B : V \to M^*$ is invertible, then:

- From $Bu = g$, we directly determine u,
- Substitute into $Au + B^*\lambda = f$ to obtain λ,
- Hence existence and uniqueness follow immediately.

Conclusion: Invertibility of B would make the problem trivial.

Realistic Case: Nontrivial Kernel

In applications, B is usually not invertible.

Define the kernel:

$$\ker B = \{u \in V \ : \ b(u,\mu) = 0, \ \forall \mu \in M\}.$$

Example:

$$Bu = \nabla \cdot u \quad \Rightarrow \quad \ker B = \{\text{divergence-free fields}\}.$$

Hence:

- u is only determined up to $\ker B$,
- uniqueness must be studied carefully.

Solvability Mechanism

From the system:

$$Au + B^*\lambda = f, \qquad Bu = g,$$

we observe:

- u must satisfy compatibility with g,
- uniqueness of u requires control on $\ker B$,
- existence of λ depends on B^*.

Key structural requirements:

- A injective on $\ker B$,
- B^* surjective onto V^*-compatible subspace.

Key Theoretical Tool

These conditions are naturally expressed by the:

Banach–Nečas–Babuška (BNB) theorem.

It replaces invertibility with an **inf-sup condition**, ensuring:

- existence of solution,
- uniqueness of solution,
- stability estimate.

This is the fundamental framework for mixed finite element methods.

From BNB to Saddle Point Problems

We first recall the variational framework:

Let V, M be reflexive Banach spaces.

Given bilinear forms:

$$a : V \times V \to \mathbb{R}, \qquad b : V \times M \to \mathbb{R},$$

we consider the saddle point problem:

$$\begin{cases} a(u, v) + b(v, \lambda) = \langle f, v\rangle, & \forall v \in V, \\ b(u, \mu) = \langle g, \mu\rangle, & \forall \mu \in M. \end{cases}$$

Goal: extend Theorem 8.1 (BNB) to this coupled system.

Recall: Banach–Nečas–Babuška (Theorem 8.1)

Let $a : U \times V \to \mathbb{R}$, $F \in V^*$.

Assume:

- **Inf-sup condition:**

$$\inf_{u \in U} \sup_{v \in V} \frac{a(u,v)}{\|u\|_U \|v\|_V} \geq c_1 > 0$$

- **Continuity:**

$$|a(u,v)| \leq c_2 \|u\|_U \|v\|_V, \quad |F(v)| \leq c_3 \|v\|_V$$

- **Injectivity:**

$$a(u,v) = 0 \ \forall u \quad \Longrightarrow \quad v = 0$$

Conclusion: there exists a unique solution with stability estimate.

Brezzi Splitting Theorem

Theorem (Brezzi splitting theorem)

Assume that

1. $a : V \times V \to \mathbb{R}$ *satisfies the conditions of Theorem 8.1 for* $U = V = \ker B$ *and*
2. $b : V \times M \to \mathbb{R}$ *satisfies for* $\beta > 0$ *the condition*

$$\inf_{\mu \in M} \sup_{v \in V} \frac{b(v, \mu)}{\|v\|_V \|\mu\|_M} \geq \beta.$$

Then there exists a unique solution $(u, \lambda) \in V \times M$ *to* $(\mathcal{S})$ *satisfying*

$$\|u\|_V + \|\lambda\|_M \leq C \left(\|f\|_{V^*} + \|g\|_{M^*} \right).$$

Condition (ii) is an inf-sup condition for B^* (since the infimum is taken over the test functions μ) and is known as the Ladyzhenskaya-Babuška-Brezzi (LBB) condition. Note that a only has to satisfy an inf-sup condition on the null space of B, not on all of V, which is crucial in many applications.

Galerkin Approximation of Saddle Point Problems

Setting. Let V and M be Hilbert spaces, and consider the mixed/saddle point problem: find $(u, \lambda) \in V \times M$ such that

$$\begin{cases} a(u, v) + b(v, \lambda) = \langle f, v \rangle_{V^*, V}, & \forall v \in V, \\ b(u, \mu) = \langle g, \mu \rangle_{M^*, M}, & \forall \mu \in M. \end{cases}$$

To construct a finite-dimensional approximation, choose subspaces

$$V_h \subset V, \qquad M_h \subset M,$$

and seek $(u_h, \lambda_h) \in V_h \times M_h$ satisfying

$$(\mathcal{S}_h) \qquad \begin{cases} a(u_h, v_h) + b(v_h, \lambda_h) = \langle f, v_h \rangle_{V^*, V}, & \forall v_h \in V_h, \\ b(u_h, \mu_h) = \langle g, \mu_h \rangle_{M^*, M}, & \forall \mu_h \in M_h. \end{cases}$$

This is the **Galerkin approximation** of the saddle point problem and is also called a **mixed finite element method**.

Why Compatibility Is Needed

The spaces V_h and M_h cannot be chosen independently. In saddle point problems, the discrete spaces must satisfy a compatibility condition in order to guarantee stability and solvability.

A key point is that even if

$$V_h \subset V, \qquad M_h \subset M,$$

the discrete operator induced by the bilinear form $b(\cdot,\cdot)$ may not behave as the restriction of the continuous operator. In particular:

- the image of V_h under B may not lie in M_h^*,
- the discrete kernel $\ker B_h$ may not coincide with $\ker B \cap V_h$,
- the discrete inf-sup condition does **not** automatically follow from the continuous one.

Therefore, stability of the method depends on a suitable pairing of the discrete spaces.

Discrete Operators and Stability Conditions

Define the discrete operator $B_h : V_h \to M_h^*$ analogously to the continuous operator B. The discrete saddle point problem is stable if the following two conditions hold.

Theorem

Assume there exist constants $\alpha_h, \beta_h > 0$ such that

$$\inf_{u_h \in \ker B_h} \sup_{v_h \in \ker B_h} \frac{a\left(u_h, v_h\right)}{\left\|u_h\right\|_V \left\|v_h\right\|_V} \geq \alpha_h$$

$$\inf_{\mu_h \in M_h} \sup_{v_h \in V_h} \frac{b\left(v_h, \mu_h\right)}{\left\|v_h\right\|_V \left\|\mu_h\right\|_M} \geq \beta_h$$

Then there exists a unique solution $(u_h, \lambda_h) \in V_h \times M_h$ to $(\mathcal{S}_h)$ satisfying

$$\left\|u_h\right\|_V + \left\|\lambda_h\right\|_M \leq C(h) \left(\|f\|_{V^*} + \|g\|_{M^*}\right).$$

Interpretation. The coercivity condition on $\ker B_h$ controls the primal variable on the discrete constraint space, while the inf-sup condition guarantees that the multiplier space M_h couples correctly with V_h.

Proof Idea for Discrete Well-Posedness

The proof follows directly from the abstract saddle point theory in finite dimensions.

Because V_h and M_h are finite-dimensional, the discrete version of the Banach–Nečas–Babuška theorem applies once the two stability conditions are available.

The key idea is:

- the coercivity of $a(\cdot,\cdot)$ on $\ker B_h$ prevents degeneracy in the primal part,
- the discrete inf-sup condition for $b(\cdot,\cdot)$ guarantees the solvability of the constraint equation,
- together they imply existence, uniqueness, and stability of the discrete pair (u_h, λ_h).

Thus, the finite element approximation is stable only if the discrete spaces are chosen in a compatible way.

Nonconforming Nature of the Discrete Approximation

Even though $V_h \subset V$ and $M_h \subset M$, the method is generally **nonconforming** in the saddle point sense.

The reason is that:

$$B(V_h) \not\subset M_h^*$$

in general, so the discrete operator B_h is not simply the restriction of B.

Moreover, one may have

$$\ker B_h \not\subset \ker B,$$

so the discrete kernel can be larger or structurally different from the continuous kernel.

As a result, the discrete inf-sup condition must be verified independently. This is one of the central issues in mixed finite element analysis.

Fortin Criterion

Theorem (Fortin criterion). Assume the continuous LBB condition holds. Then the discrete LBB condition is satisfied if and only if there exists a linear operator

$$\Pi_h : V \to V_h$$

such that

$$b(\Pi_h v, \mu_h) = b(v, \mu_h) \qquad \forall \mu_h \in M_h,$$

and

$$\|\Pi_h v\|_V \le \gamma_h \|v\|_V \qquad \forall v \in V,$$

for some constant $\gamma_h > 0$.

Meaning. The operator Π_h preserves the action of $b(\cdot,\cdot)$ on the discrete multiplier space. Such an operator is called a **Fortin projector**.

If γ_h can be chosen independently of h, then the discrete inf-sup constant is also independent of h.

Proof.

Assume that such a Π_h exists. Since $\operatorname{ran}\Pi_h \subset V_h$, we have for all $\mu_h \in M_h \subset M$ that

$$\sup_{v_h \in V_h} \frac{b\left(v_h, \mu_h\right)}{\left\|v_h\right\|_V} \geq \sup_{v \in V} \frac{b\left(\Pi_h v, \mu_h\right)}{\left\|\Pi_h v\right\|_V} \geq \sup_{v \in V} \frac{b\left(v, \mu_h\right)}{\gamma_h \|v\|_V} \geq \frac{\beta}{\gamma_h}\left\|\mu_h\right\|_M,$$

which implies the discrete LBB condition. Conversely, if the discrete LBB condition holds, the operator $B_h : V_h \to M_h^*$ as defined above is surjective and has a continuous right inverse. Furthermore, for any $v \in V$ we can consider $Bv \in M^*$ as a linear functional on $M_h \subset M$ only. Hence for any $v \in V$, there exists a $\pi_h \in V_h$ such that $B_h\left(\pi_h\right) = \left.Bv\right|_{M_h} \in M_h^*$, i.e., $b\left(\pi_h, \mu_h\right) = b\left(v, \mu_h\right)$ for all $\mu_h \in M_h \subset M$, and

$$\beta_h \left\|\pi_h\right\|_V \leq \|Bv\|_{M_h^*} \leq C\|v\|_V$$

We thus obtain the desired operator by defining Π_h as the (linear) mapping $v \mapsto \pi_h$. □

A Priori Error Estimate

Céa-type estimate for saddle point problems.

Theorem

Assume that the discrete stability conditions hold, i.e., there exist constants $\alpha_h, \beta_h > 0$ *such that*

$$\inf_{u_h \in \ker B_h} \sup_{v_h \in \ker B_h} \frac{a(u_h, v_h)}{\|u_h\|_V \, \|v_h\|_V} \geq \alpha_h, \qquad \inf_{\mu_h \in M_h} \sup_{v_h \in V_h} \frac{b(v_h, \mu_h)}{\|v_h\|_V \, \|\mu_h\|_M} \geq \beta_h.$$

Let $(u, \lambda) \in V \times M$ *be the exact solution and* $(u_h, \lambda_h) \in V_h \times M_h$ *be the discrete solution. Then there exists a constant* $C(h) > 0$ *such that*

$$\|u - u_h\|_V + \|\lambda - \lambda_h\|_M \leq C(h) \left(\inf_{w_h \in V_h} \|u - w_h\|_V + \inf_{\mu_h \in M_h} \|\lambda - \mu_h\|_M \right).$$

Interpretation. The total error is controlled by the best approximation errors of u and λ in the discrete spaces V_h and M_h.

This result is the saddle point analogue of Céa' s lemma: stability (inf-sup + coercivity) together with approximation properties completely determine the convergence rate.

Error Estimate for the Primal Variable

If the discrete kernel satisfies

$$\ker B_h \subset \ker B,$$

then the primal error can be estimated independently of the multiplier.

Corollary

If $\ker B_h \subset \ker B$, *then there exists a constant* $C(h) > 0$ *such that*

$$\|u - u_h\|_V \leq C(h) \inf_{w_h \in V_h} \|u - w_h\|_V .$$

This result is especially useful when one is mainly interested in the primary unknown u.

Summary

1. 混合格式

同时近似主变量 u 与约束变量 λ，适合约束问题、不可压问题和鞍点结构问题。

2. 稳定性

离散 inf-sup 条件是混合元可解性和误差分析的核心；没有稳定性，就没有可靠的离散解。

3. Fortin 投影

它把“连续稳定性”传递到“离散稳定性”，是构造稳定元对的关键工具。

4. 误差估计

最终误差由最佳逼近误差控制；若核空间嵌入成立，还能得到独立的主变量误差估计。

Stable discretization $\Longrightarrow$ well-posedness $\Longrightarrow$ convergence.

Mixed Methods for the Poisson Equation

Model problem. Consider the Poisson equation with homogeneous Dirichlet boundary condition

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

To derive a mixed formulation, introduce the flux variable $\sigma = \nabla u$. Then the equation can be rewritten as the first-order system

$$\begin{cases} \nabla u - \sigma = 0, \\ -\nabla \cdot \sigma = f. \end{cases}$$

This system can be formulated in two different variational ways:

- **primal mixed method**, obtained by integrating by parts in the second equation;
- **dual mixed method**, obtained by integrating by parts in the first equation.

These two formulations illustrate the general mixed finite element theory in a very simple setting.

Primal Mixed Method: Variational Formulation

In the primal approach, integrate by parts in the equation

$$-\nabla \cdot \sigma = f.$$

We look for $(\sigma, u) \in L^2(\Omega)^n \times H_0^1(\Omega)$ such that

$$\begin{cases} (\sigma, \tau) - (\tau, \nabla u) = 0, & \forall \tau \in L^2(\Omega)^n, \\ -(\sigma, \nabla v) = -(f, v), & \forall v \in H_0^1(\Omega). \end{cases}$$

This fits the abstract saddle point framework with

$$V := L^2(\Omega)^n, \qquad M := H_0^1(\Omega),$$

and bilinear forms

$$a(\sigma, \tau) = (\sigma, \tau), \qquad b(\tau, v) = -(\tau, \nabla v).$$

The role of the unknowns is: ■ σ is the flux variable in $L^2(\Omega)^n$, ■ u is the scalar potential in $H_0^1(\Omega)$.

Kernel of B in the Primal Mixed Method

For the primal mixed formulation of the Poisson equation,

$$V = L^2(\Omega)^n, \qquad M = H_0^1(\Omega), \qquad b(\tau, v) = -(\tau, \nabla v),$$

the operator $B : V \to M^*$ is defined by

$$\langle B\tau, v\rangle = b(\tau, v) = -(\tau, \nabla v).$$

$$\ker B = \left\{\tau \in L^2(\Omega)^n : \ (\tau, \nabla v) = 0, \ \forall v \in H_0^1(\Omega)\right\}.$$

Equivalent characterization. Since

$$(\tau, \nabla v) = -(\operatorname{div}\tau, v) \quad \text{(in weak sense)},$$

we obtain

$$\ker B = \left\{\tau \in L^2(\Omega)^n : \ \operatorname{div}\tau = 0 \text{ (in distribution sense)}\right\}.$$

Geometric interpretation.

$$\ker B = \left(\nabla H_0^1(\Omega)\right)^{\perp} \subset L^2(\Omega)^n,$$

i.e., the space of vector fields that are orthogonal to all gradient fields.

Primal Mixed Method: Stability

The bilinear form $a(\cdot,\cdot)$ is coercive on the whole space $V = L^2(\Omega)^n$:

$$a(\sigma,\sigma) = \|\sigma\|^2_{L^2(\Omega)^n},$$

so the coercivity constant is $\alpha = 1$.

To verify the LBB condition, let $v \in H^1_0(\Omega)$. Choosing $\tau = -\nabla v \in L^2(\Omega)^n = V$, gives

$$\sup_{\tau\in V}\frac{b(\tau,v)}{\|\tau\|_V} = \sup_{\tau\in V}\frac{-(\tau,\nabla v)}{\|\tau\|_{L^2(\Omega)^n}} \geq \frac{(\nabla v,\nabla v)}{\|\nabla v\|_{L^2(\Omega)^n}} = |v|_{H^1(\Omega)}.$$

Using the Poincaré inequality,

$$|v|_{H^1(\Omega)} \geq c_\Omega^{-1}\|v\|_{H^1_0(\Omega)},$$

hence the LBB constant is $\beta = c_\Omega^{-1}$.

Therefore the abstract saddle point theory yields existence and uniqueness of the solution.

Primal Mixed Finite Element Method

Choose a shape-regular affine triangulation $\mathcal{T}_h$ of Ω. For $k \geq 1$, define

$$V_h := \left\{ \tau_h \in L^2(\Omega)^n : \ \tau_h|_K \in P_{k-1}(K)^n, \ \forall K \in \mathcal{T}_h \right\},$$

$$M_h := \left\{ v_h \in C^0(\Omega) : \ v_h|_K \in P_k(K), \ \forall K \in \mathcal{T}_h \right\} \cap H_0^1(\Omega).$$

Then

$$V_h \subset V, \qquad M_h \subset M.$$

Since $a(\cdot,\cdot)$ is the L^2 inner product, coercivity on V_h is immediate.

Moreover,

$$\nabla M_h \subset V_h,$$

because the gradient of a continuous piecewise polynomial of degree k is piecewise polynomial of degree $k-1$.

Hence the L^2 projection $\Pi_h : V \to V_h$ satisfies that given $\sigma \in V$,

$$b(\Pi_h \sigma, v_h) = -(\Pi_h \sigma, \nabla v_n) = -(\sigma, \nabla v_n) = b(\sigma, v_h) \qquad \forall v_h \in M_h, \quad (\nabla v_h \in V_h),$$

so the Fortin criterion holds and the discrete LBB constant is independent of h.

Primal Mixed Method: Consequences

Since the Fortin criterion is satisfied, the discrete inf-sup constant is uniform in h:

$$\beta_h \geq \beta > 0.$$

Therefore the primal mixed finite element method is stable and admits a unique solution

$$(\sigma_h, u_h) \in V_h \times M_h.$$

Combining the discrete stability with the standard interpolation estimates gives a priori error bounds of the form

$$\|\sigma - \sigma_h\|_{L^2(\Omega)^n} + \|u - u_h\|_{H^1(\Omega)} \leq Ch^k \left(\|\sigma\|_{H^k(\Omega)} + \|u\|_{H^{k+1}(\Omega)}\right),$$

under the usual regularity assumptions.

This example shows how the abstract theory translates into a concrete and stable mixed finite element scheme.

Dual Mixed Method: First-Order System

Instead of integrating by parts in the second equation, we integrate by parts in

$$\nabla u - \sigma = 0.$$

To make this meaningful, we introduce the space

$$H^{\operatorname{div}}(\Omega) := \left\{ \tau \in L^2(\Omega)^n : \ \operatorname{div} \tau \in L^2(\Omega) \right\},$$

with graph norm

$$\|\tau\|^2_{H^{\operatorname{div}}(\Omega)} = \|\tau\|^2_{L^2(\Omega)^n} + \|\operatorname{div} \tau\|^2_{L^2(\Omega)}.$$

$\tau \in H^{\operatorname{div}}(\Omega) \subset L^2(\Omega)$ has a well-defined **normal trace** $(\tau|_{\partial\Omega} \cdot \nu) \in H^{-\frac{1}{2}}(\partial\Omega)$. For $\tau \in H^{\operatorname{div}}(\Omega)$ and $w \in H^1(\Omega)$, the integration by parts formula holds:

$$\int_\Omega (\operatorname{div} \tau)\, w \, dx + \int_\Omega \tau \cdot \nabla w \, dx = \int_{\partial\Omega} (\tau \cdot \nu)\, w \, ds,$$

where ν is the outward unit normal.

A key point is that piecewise smooth vector fields with continuous normal traces across interfaces belong to $H^{\operatorname{div}}(\Omega)$.

Dual Mixed Method: Variational Formulation

Using the boundary condition $u|_{\partial\Omega} = 0$ and integrating by parts in the first equation, we look for $(\sigma, u) \in H^{\mathrm{div}}(\Omega) \times L^2(\Omega)$ such that

$$\begin{cases} (\sigma, \tau) + (\operatorname{div} \tau, u) = 0, & \forall \tau \in H^{\mathrm{div}}(\Omega), \\ (\operatorname{div} \sigma, v) = -(f, v), & \forall v \in L^2(\Omega). \end{cases}$$

This is again a saddle point problem with

$$V := H^{\mathrm{div}}(\Omega), \qquad M := L^2(\Omega),$$

and

$$a(\sigma, \tau) = (\sigma, \tau), \qquad b(\tau, v) = (\operatorname{div} \tau, v).$$

Notice that the Dirichlet condition ($u|_{\partial\Omega} = 0$) now appears as a natural boundary condition in the weak formulation.

$$\begin{aligned} -(\nabla u - \sigma, \tau) =& (\sigma, \tau) - (\nabla u, \tau) = (\sigma, \tau) + (\operatorname{div} \tau, u) - \int_{\partial\Omega} (\tau \cdot \nu)\, u \, ds, \\ =& (\sigma, \tau) + (\operatorname{div} \tau, u) \end{aligned}$$

Dual Mixed Method: Coercivity and LBB Condition

The kernel of the operator B is

$$\ker B = \left\{\tau \in H^{\mathrm{div}}(\Omega) : (\operatorname{div}\tau, v) = 0 \ \forall v \in L^2(\Omega)\right\}.$$

Since $\operatorname{div}\tau \in L^2(\Omega)$ and thus $\|\operatorname{div}\tau\|^2_{L^2(\Omega)} = 0$, it follows that

$$\operatorname{div}\tau = 0 \qquad \text{for all } \tau \in \ker B.$$

Hence, on $\ker B$,

$$a(\tau,\tau) = \|\tau\|^2_{L^2(\Omega)^n} = \|\tau\|^2_{H^{\mathrm{div}}(\Omega)},$$

so a is coercive with constant $\alpha = 1$.

To prove the LBB condition, one needs the surjectivity of

$$B : H^{\mathrm{div}}(\Omega) \to L^2(\Omega)^*.$$

This is typically established by constructing a vector field with prescribed divergence, under suitable assumptions on Ω such as convexity or C^1 boundary regularity.

A Right Inverse of the Divergence Operator

Lemma (Lemma 10.6)

For any $f \in L^2(\Omega)$*, there exists a function* $\tau \in H^1(\Omega)^n$ *with* $\operatorname{div} \tau = f$ *and* $\|\tau\|_{H^1(\Omega)^n} \le C\|f\|_{L^2(\Omega)}$.

Proof.

Due to the regularity of Ω, we can apply elliptic regularity theory. For given $f \in L^2(\Omega)$, there exists a solution $u \in H^2(\Omega) \cap H_0^1(\Omega)$ to the Poisson equation

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega),$$

satisfying the estimate $\|u\|_{H^2(\Omega)} \le C\|f\|_{L^2(\Omega)}$. Now define $\tau := -\nabla u$. Since $u \in H^2(\Omega)$, we obtain $\tau \in H^1(\Omega)^n$. Moreover,

$$(f, v) = -(\tau, \nabla v) \qquad \forall v \in H_0^1(\Omega).$$

Hence, by the definition of weak divergence, $\operatorname{div} \tau = f$. The a priori bound on τ then follows from the fact that $\|\nabla u\|_{H^1(\Omega)^n} \le \|u\|_{H^2(\Omega)}$. □

Verification of the LBB Condition

Using Lemma 10.6, let $v \in M = L^2(\Omega)$. There exists $\tau_v \in H^1(\Omega)^n \subset H^{\mathrm{div}}(\Omega)$ such that $\operatorname{div} \tau_v = v$ and $\|\tau_v\|_{H^1(\Omega)^n} \leq C\|v\|_{L^2(\Omega)}$.

Then

$$\begin{aligned}\sup_{\tau \in V} \frac{b(\tau, v)}{\|\tau\|_V} &= \sup_{\tau \in V} \frac{(\operatorname{div} \tau, v)}{\|\tau\|_{H^{\mathrm{div}}(\Omega)}} \geq \frac{(\operatorname{div} \tau_v, v)}{\|\tau_v\|_{H^{\mathrm{div}}(\Omega)}} \\ &= \frac{(v, v)}{\|\tau_v\|_{H^{\mathrm{div}}(\Omega)}} \geq \frac{(v, v)}{\|\tau_v\|_{H^1(\Omega)}} \geq \frac{\|v\|^2_{L^2(\Omega)}}{C\|v\|_{L^2(\Omega)}} = \frac{1}{C}\|v\|_{L^2(\Omega)}.\end{aligned}$$

Therefore,

$$\inf_{v \in M} \sup_{\tau \in V} \frac{b(\tau, v)}{\|\tau\|_V \|v\|_M} \geq \frac{1}{C}.$$

Hence the LBB condition holds with $\beta = C^{-1}$.

Well-Posedness of the Mixed Problem

Therefore, we obtain that the mixed formulation admits a unique solution $(\sigma, u) \in H^{\mathrm{div}}(\Omega) \times L^2(\Omega)$. Moreover,

$$\|\sigma\|_{H^{\mathrm{div}}(\Omega)} + \|u\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}.$$

Initially, the mixed theory only gives $u \in L^2(\Omega)$. However, using the first mixed equation,

$$(\sigma, \tau) + (\operatorname{div} \tau, u) = 0,$$

one can show:

- u possesses weak derivatives,
- integration by parts recovers the boundary condition,
- therefore $u \in H_0^1(\Omega)$.

Thus the mixed formulation is equivalent to the classical weak formulation of Poisson's equation.

Finite Element Discretization

We now construct conforming finite element discretizations of V and M. Let $\mathcal{T}_h$ be a shape-regular affine triangulation of $\Omega \subset \mathbb{R}^n$. We seek conforming finite element spaces

$$V_h \subset H^{\mathrm{div}}(\Omega), \qquad M_h \subset L^2(\Omega).$$

For the scalar variable, choose discontinuous piecewise polynomials:

$$M_h = \left\{ v_h \in L^2(\Omega) : v_h|_K \in P_k(K) \quad \forall K \in \mathcal{T}_h \right\}.$$

For the flux variable, we need a space satisfying:

- continuity of normal components across interfaces,
- surjectivity of the discrete divergence operator, $\operatorname{div} V_h = M_h$.

The classical choice is the Raviart–Thomas space.

Raviart–Thomas Finite Elements

Definition (Raviart–Thomas Space)

Define

$$V_h = \left\{\tau_h \in H^{\mathrm{div}}(\Omega) : \tau_h|_K \in RT_k(K) \quad \forall K \in \mathcal{T}_h\right\},$$

where

$$RT_k(K) = P_k(K)^n + xP_k(K) = RT_k(K) = P_k(K)^n \oplus xP_k^0(K),$$

where

$$P_k^0(K) = \left\{\sum_{|\alpha|=k} c_\alpha x^\alpha : c_\alpha \in \mathbb{R}\right\}$$

is the space of homogeneous polynomials of degree k.

Thus every function in $RT_k(K)$ has the form

$$\tau_h(x) = p_1(x) + x\,p_2(x),$$

with $p_1 \in P_k(K)^n$, $p_2 \in P_k(K)$.

Properties of Raviart–Thomas Elements

Lemma (Lemma 10.7)

For $\tau_h \in RT_k(K)$, *we have:*

1. $\operatorname{div} \tau_h \in P_k(K)$,
2. *for every face* $F \subset \partial K$, $\tau_h|_F \cdot \nu_F \in P_k(F)$.

Explanation.

- Property (i) guarantees compatibility with $M_h = P_k(K)$.
- Property (ii) guarantees continuity of normal traces across interfaces.
- Therefore, $V_h \subset H^{\mathrm{div}}(\Omega)$.

Recall: for any face F, $x \cdot \nu_F(x)$ is constant on F, which simplifies the proof.

Dimension of the Raviart–Thomas Space

The Raviart–Thomas space has dimension

$$\dim RT_k(K) = \begin{cases} (k+1)(k+3), & n = 2, \\ \dfrac{1}{2}(k+1)(k+2)(k+4), & n = 3. \end{cases}$$

To construct a finite element, we must specify exactly this number of degrees of freedom.

The degrees of freedom are divided into:

- face moments,
- interior moments.

The face moments enforce continuity of normal components across neighboring elements, which is the key requirement for

$H^{\mathrm{div}}(\Omega)$-conformity.

Degrees of Freedom of the Raviart–Thomas Element

For each face $F_i \subset \partial K$, define

$$N_{i,j}(\tau) = \int_{F_i} (\tau \cdot \nu_i)\, q_{ij}\, ds,$$

where $\{q_{ij}\}$ is a basis of $P_k(F_i)$.

If $k \geq 1$, define additional interior moments:

$$N_{0,j}(\tau) = \int_K \tau \cdot q_j\, dx,$$

where $\{q_j\}$ is a basis of $P_{k-1}(K)^n$.

Interpretation.

- Face moments determine the normal trace.
- Interior moments determine the remaining tangential information.
- Together they uniquely determine the polynomial vector field.

Unisolvence of the Raviart–Thomas Element

Lemma

If $\tau_h \in RT_k(K)$ *satisfies* $N_{i,j}(\tau_h) = 0$ *for all* i, j, *then* $\tau_h = 0$.

Let $K = \{(x,y) : x \geq 0,\ y \geq 0,\ x + y \leq 1\}$ be the reference triangle.

$RT_0(K)$ **element.**

Since $RT_0(K) = P_0(K)^2 + \boldsymbol{x}P_0(K)$, every function has the form

$$\tau(x,y) = \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix} c = \begin{pmatrix} a + cx \\ b + cy \end{pmatrix}.$$

Its divergence is

$$\operatorname{div} \tau = 2c \in P_0(K).$$

Degrees of freedom:

$$N_i(\tau) = \int_{F_i} \tau \cdot \nu_i \, ds, \qquad i = 1, 2, 3.$$

Hence $\dim RT_0(K) = 3$.

$RT_1(K)$ Element

Since $RT_1(K) = P_1(K)^2 + \boldsymbol{x}P_1(K)$, a general function can be written as

$$\tau(x, y) = \begin{pmatrix} a_1 + a_2 x + a_3 y \\ b_1 + b_2 x + b_3 y \end{pmatrix} + \begin{pmatrix} x(c_1 + c_2 x + c_3 y) \\ y(c_1 + c_2 x + c_3 y) \end{pmatrix}.$$

Its divergence satisfies

$$\operatorname{div} \tau \in P_1(K).$$

Degrees of freedom:

- Edge moments:

  $$\int_F (\tau \cdot \nu) q \, ds, \qquad q \in P_1(F).$$

  Each edge contributes 2 DOFs.

- Interior moments:

  $$\int_K \tau \cdot q \, dx, \qquad q \in P_0(K)^2.$$

  There are 2 interior DOFs.

Therefore, $\dim RT_1(K) = 3 \times 2 + 2 = 8$.

$$RT_1(K) = P_1(K)^2 + xP_1(K).$$

$$p_1 = \begin{pmatrix} a_1 + a_2x + a_3y \\ b_1 + b_2x + b_3y \end{pmatrix}, \qquad p_2 = c_1 + c_2x + c_3y.$$

Then $\tau = p_1 + x\,p_2$ that is,

$$\tau(x,y) = \begin{pmatrix} a_1 + a_2x + a_3y + c_1x + c_2x^2 + c_3xy \\ b_1 + b_2x + b_3y + c_1y + c_2xy + c_3y^2 \end{pmatrix}.$$

After rearranging,

$$\tau(x,y) = \begin{pmatrix} a_1 + d_1x + a_3y + c_2x^2 + c_3xy \\ b_1 + b_2x + d_2y + c_2xy + c_3y^2 \end{pmatrix},$$

where $d_1 = a_2 + c_1$, $d_2 = b_3 + c_1$.

Hence there remain only 8 independent coefficients:

$a_1,\ d_1,\ a_3,\ c_2,\ c_3,\ b_1,\ b_2,\ d_2.$

Therefore, $\dim RT_1(K) = 8$. Equivalently, a basis of $RT_1(K)$ can be written as

$$RT_1(K) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} y \\ 0 \end{pmatrix}, \begin{pmatrix} x^2 \\ xy \end{pmatrix}, \begin{pmatrix} xy \\ y^2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ x \end{pmatrix}, \begin{pmatrix} 0 \\ y \end{pmatrix} \right\}.$$

Summary: Primal vs. Dual Mixed Methods

Primal mixed method

- Spaces: $V = L^2(\Omega)^n$, $M = H_0^1(\Omega)$,
- Flux variable in L^2,
- Gradient coupling through $b(\tau, v) = -(\tau, \nabla v)$,
- Stable discrete spaces can be built using standard C^0 finite elements for u.

Dual mixed method

- Spaces: $V = H^{\mathrm{div}}(\Omega)$, $M = L^2(\Omega)$,
- Flux variable has square-integrable divergence,
- Divergence coupling through $b(\tau, v) = (\operatorname{div} \tau, v)$,
- Conforming approximations require $H(\mathrm{div})$-conforming elements.

Both formulations are special cases of the abstract saddle point theory:

$$\text{coercivity on } \ker B \;+\; \text{LBB condition} \;\Longrightarrow\; \text{well-posed mixed method.}$$

Mixed Formulation of the Stokes Equation

Consider the Stokes problem:

$$\begin{aligned} -\Delta \boldsymbol{u} + \nabla p &= \boldsymbol{f} && \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} &= 0 && \text{in } \Omega, \end{aligned}$$

with boundary condition $\boldsymbol{u} = 0 \quad$ on $\partial\Omega$.

Here $\boldsymbol{u}:$ velocity, $p:$ pressure.

Define

$$\boldsymbol{V} = H_0^1(\Omega)^2, \quad M = L_0^2(\Omega) = \left\{ q \in L^2(\Omega) : \int_\Omega q \, d\boldsymbol{x} = 0 \right\}.$$

The weak mixed formulation reads:

Find $(\boldsymbol{u}, p) \in \boldsymbol{V} \times M$ such that

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) &= (\boldsymbol{f}, \boldsymbol{v}) && \forall \boldsymbol{v} \in \boldsymbol{V}, \\ b(\boldsymbol{u}, q) &= 0 && \forall q \in M, \end{aligned}$$

where $a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v} \, d\boldsymbol{x}$, $b(\boldsymbol{v}, q) = -(q, \operatorname{div} \boldsymbol{v})$.

Taylor–Hood Finite Element Pair

A classical stable mixed finite element pair for the Stokes equation is the Taylor–Hood element.

Choose

$$\boldsymbol{V}_h = \left\{ v_h \in H_0^1(\Omega)^2 : v_h|_K \in P_2(K)^2 \right\},$$

and

$$M_h = \left\{ q_h \in L_0^2(\Omega) : q_h|_K \in P_1(K) \right\}.$$

That is, • **quadratic** velocity approximation, • **linear** pressure approximation.

The discrete problem becomes:

Find $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times M_h$ such that

$$\begin{aligned} & a(\boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, p_h) = (f, \boldsymbol{v}_h) && \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ & b(\boldsymbol{u}_h, q_h) = 0 && \forall q_h \in M_h. \end{aligned}$$