有限元方法

Finite Element Methods

Chapter 5: Polynomial Interpolation In Sobolev Spaces

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Outline

1. POLYNOMIAL INTERPOLATION IN SOBOLEV SPACES
   - The Bramble-Hilbert Lemma
   - Interpolation Error Estimates
   - Inverse Estimates
   - Error Estimates for the Finite Element Approximation

Interpolation Error: The Core of FEM Theory

Key Idea: Error Decomposition

- The accuracy of the finite element solution depends on:

$$\|u - u_h\|$$

- This error is controlled by the **best approximation error**:

$$\|u - u_h\| \lesssim \inf_{v_h \in V_h} \|u - v_h\|$$

- The best approximation is further bounded by the **interpolation error**:

$$\inf_{v_h \in V_h} \|u - v_h\| \leq \|u - I_h u\|$$

Conclusion

- The FEM error analysis reduces to estimating:

$$\|u - I_h u\|$$

- This leads to the study of:
  - **Local interpolation error**
  - **Global interpolation error**

We start with the error for the local interpolant. The key for deriving error estimates is the Bramble-Hilbert lemma. The derivation here follows the original functional-analytic arguments (by way of several results which may be of independent interest); there are also constructive approaches which allow more explicit computation of the constants.

The first lemma characterizes the kernel of differentiation operators.

Lemma

If $v \in W^{k,p}(\Omega)$ satisfies $D^\alpha v = 0$ for all $|\alpha| = k$, then v is almost everywhere equal to a polynomial of degree $k-1$.

The next result concerns moment interpolation of Sobolev functions on polynomials.

Lemma

For every $v \in W^{k,p}(\Omega)$ there is a unique polynomial $q \in P_{k-1}$ such that

$$\int_\Omega D^\alpha(v-q)dx = 0 \quad \text{for all } |\alpha| \le k-1.$$

Poincaré's inequality

The last lemma is a generalization of Poincaré's inequality.

Lemma

Let $v \in W^{k,p}(\Omega)$ *such that*

$$\int_\Omega D^\alpha v dx = 0 \quad \textit{for all } |\alpha| \le k-1$$

Then

$$\|v\|_{W^{k,p}(\Omega)} \le c_0 |v|_{W^{k,p}(\Omega)}$$

where the constant $c_0 > 0$ *depends only on* Ω, k *and* p.

Bramble-Hilbert lemma

Theorem (Bramble-Hilbert lemma)

Let $F : W^{k,p}(\Omega) \to \mathbb{R}$ *satisfy*

1. $|F(v)| \leq c_1 \|v\|_{W^{k,p}(\Omega)}$ *for all* $v \in W^{k,p}(\Omega)$ *(boundedness),*
2. $|F(u+v)| \leq c_2(|F(u)| + |F(v)|)$ *for all* $u, v \in W^{k,p}(\Omega)$ *(sublinearity),*
3. $F(q) = 0$ *for all* $q \in P_{k-1}$ *(annihilation).*

Then there exists a constant $c > 0$ *such that for all* $v \in W^{k,p}(\Omega)$*,*

$$|F(v)| \leq c|v|_{W^{k,p}(\Omega)}$$

Bramble–Hilbert Lemma: Intuition

Goal: Control interpolation / approximation error

Core Principle

If an error operator cannot "see" low-degree polynomials, then the error depends only on **high-order derivatives**.

Three Essential Properties of F

1. **Boundedness**
   Error is controlled by the full Sobolev norm.
2. **Sublinearity**
   Error behaves well under decomposition.
3. **Annihilation**
   $F(q) = 0$ for all polynomials $q \in P_{k-1}$
   (exact for low-order polynomials)

Bramble–Hilbert Lemma (Conclusion)

$$|F(v)| \le C\ |v|_{W^{k,p}(\Omega)}$$

Error is controlled by the **k-th order derivatives** only.

Interpretation

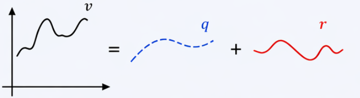
- Decompose: $v =$ polynomial part + high-order remainder
- F kills the polynomial part $\Rightarrow$ only the remainder matters
- Error $\sim$ **high-order derivatives**

Takeaway: If an error operator is exact on polynomials, it only measures what polynomials cannot represent.

Visual Explanation

1. Decompose the function

$$v = q + r$$



original function

polynomial part (degree $\le k-1$)

high-order remainder (contains the fine scales)

2. Annihilation property

$F(q) = 0$

$F(v) = F(q + r) = F(r)$

F cannot "see" polynomials

F only "sees" the remainder

3. Result

Error depends only on **high-order derivatives**

$$|F(v)| \le C\ |v|_{W^{k,p}(\Omega)}$$

Smaller mesh / higher regularity $\Rightarrow$ smaller error

Big Picture: Low-order polynomials are captured exactly. The error comes solely from what lies **beyond degree $k-1$**.

Interpolation error

We wish to apply the Bramble-Hilbert lemma to the interpolation error. We start with the error on the reference element.

Theorem

Let $(K, \mathcal{P}, \mathcal{N})$ *be a finite element with* $P_{k-1} \subset \mathcal{P}$ *for some* $k \geq 1$ *and all* $N \in \mathcal{N}$ *bounded on* $W^{k,p}(K), 1 \leq p \leq \infty$. *Then for any* $v \in W^{k,p}(K)$,

$$|v - \mathcal{I}_K v|_{W^{l,p}(K)} \leq c|v|_{W^{k,p}(K)} \quad \text{for all} \quad 0 \leq l \leq k$$

where the constant $c > 0$ *depends only on* n, k, p, l *and* $(K, \mathcal{P}, \mathcal{N})$.

Proof.

It is straightforward to verify that $F : v \mapsto |v - \mathcal{I}_K v|_{W^{l,p}(K)}$ defines a sublinear functional on $W^{k,p}(K)$ for all $l \le k$. Let $\psi_1, \dots, \psi_d$ be the nodal basis of $\mathcal{P}$ to $\mathcal{N}$. Since the N_i in $\mathcal{N}$ are bounded on $W^{k,p}(K)$, we have that

$$\begin{aligned}
|F(v)| &\le |v|_{W^{l,p}(K)} + |\mathcal{I}_K v|_{W^{l,p}(K)} \\
&\le \|v\|_{W^{k,p}(K)} + \sum_{i=1}^{d} |N_i(v)| |\psi_i|_{W^{l,p}(K)} \\
&\le \|v\|_{W^{k,p}(K)} + \sum_{i=1}^{d} C_i \|v\|_{W^{k,p}(K)} |\psi_i|_{W^{l,p}(K)} \\
&\le \left(1 + C \max_{1 \le i \le d} |\psi_i|_{W^{l,p}(K)}\right) \|v\|_{W^{k,p}(K)}
\end{aligned}$$

and hence that F is bounded. In addition, $\mathcal{I}_K q = q$ for all $q \in \mathcal{P}$ and therefore $F(q) = 0$. We can now apply the Bramble-Hilbert lemma to F, which proves the claim. □

To estimate the interpolation error on an arbitrary finite element $(K, \mathcal{P}, \mathcal{N})$, we assume that it is generated by the affine transformation

$$T_K : \hat{K} \to K, \quad \hat{x} \mapsto A_K \hat{x} + b_K$$

from the reference element $(\hat{K}, \hat{\mathcal{P}}, \hat{\mathcal{N}})$, i.e., $\hat{v} := v \circ T_K$ is the function v on K expressed in local coordinates on $\hat{K}$. We then need to consider how the estimate

$$|v - \mathcal{I}_K v|_{W^{l,p}(K)} \leq c|v|_{W^{k,p}(K)} \quad \text{for all} \quad 0 \leq l \leq k$$

transforms under T_K. For this, we recall that for sufficiently smooth v, the chain rule for weak derivatives is given by

$$\frac{\partial \hat{v}}{\partial \hat{x}_i} = \sum_{j=1}^{n} \frac{\partial v}{\partial x_j} \frac{\partial x_j}{\partial \hat{x}_i} = \sum_{j=1}^{n} (A_K)_{ij} \frac{\partial v}{\partial x_j}$$

and the transformation rule for integrals by

(T) $$\int_{T_K(\hat{K})} v dx = \int_{\hat{K}} (v \circ T_K) \left|\det (A_K)\right| d\hat{x}$$

Lemma

Let $k \geq 0$ *and* $1 \leq p \leq \infty$. *There exists* $c > 0$ *such that for all* K *and* $v \in W^{k,p}(K)$, *the function* $\hat{v} = v \circ T_K$ *satisfies*

(E1) $$|\hat{v}|_{W^{k,p}(\hat{K})} \leq c \|A_K\|^k |\det(A_K)|^{-\frac{1}{p}} |v|_{W^{k,p}(K)},$$

(E2) $$|v|_{W^{k,p}(K)} \leq c \left\|A_K^{-1}\right\|^k |\det(A_K)|^{\frac{1}{p}} |\hat{v}|_{W^{k,p}(\hat{K})}.$$

Proof.

First, we have $\hat{v} \in W^{k,p}(\hat{K})$. Let now α be a multi-index with $|\alpha| = k$, and let $\hat{D}^\alpha$ denote the corresponding weak derivative with respect to $\hat{x}$. Applying the chain and transformation rule, we obtain with a constant c depending only on n, k, and p that

$$\begin{aligned} \left\|\hat{D}^\alpha \hat{v}\right\|_{L^p(\hat{K})} &\leq c \|A_K\|^k \sum_{|\beta|=k} \left\|D^\beta v \circ T_K\right\|_{L^p(\hat{K})} \\ &\leq c \|A_K\|^k |\det(A_K)|^{-\frac{1}{p}} |v|_{W^{k,p}(K)}. \end{aligned}$$

Summing over all $|\alpha| = k$ yields (E1). Arguing similarly using T_K^{-1} yields (E2). □

We now derive a geometrical estimate of the quantities appearing in the right-hand side of (E1) and (E2). For a given element domain K, we define

- the diameter $h_K := \max_{x_1, x_2 \in K} \|x_1 - x_2\|$,
- the insphere diameter $\rho_K := 2\max\{\rho > 0 : B_\rho(x) \subset K \text{ for some } x \in K\}$ (i.e., the diameter of the largest ball contained in K).
- the condition number $\sigma_K := \frac{h_K}{\rho_K}$.

Lemma

Let T_K be an affine mapping such that $K = T_K(\hat{K})$. Then

$$|\det(A_K)| = \frac{\mathrm{vol}(K)}{\mathrm{vol}(\hat{K})}, \quad \|A_K\| \leq \frac{h_K}{\rho_{\hat{K}}}, \quad \left\|A_K^{-1}\right\| \leq \frac{h_{\hat{K}}}{\rho_K}.$$

Proof.

The first property follows from the transformation rule (T) applied to the constant function $v \equiv 1$. For the second property, recall that the matrix norm of A_K is given by

$$\|A_K\| = \sup_{\|\hat{x}\|=1} \|A_K \hat{x}\| = \frac{1}{\rho_{\hat{K}}} \sup_{\|\hat{x}\|=\rho_{\hat{K}}} \|A_K \hat{x}\|.$$

Now for any $\hat{x}$ with $\|\hat{x}\| = \rho_{\hat{K}}$, there exists $\hat{x}_1, \hat{x}_2 \in \hat{K}$ with $\hat{x} = \hat{x}_1 - \hat{x}_2$ (e.g., choose a suitable $\hat{x}_1$ on the insphere and $\hat{x}_2$ as its antipodal point). Then

$$A_K \hat{x} = T_K \hat{x}_1 - T_K \hat{x}_2 = x_1 - x_2 \qquad \text{for some } x_1, x_2 \in K$$

which implies $\|A_K \hat{x}\| \leq h_K$ and thus the desired inequality. The last property is obtained by exchanging the roles of K and $\hat{K}$. □

Note that since the insphere of diameter ρ_K is contained in K, which in turn is contained in the surrounding sphere of diameter h_K, we can further estimate (with a constant c depending only on n)

$$ch_K^n \geq \text{vol}(K) \geq c\rho_K^n = c\frac{h_K^n}{\sigma_K^n}$$

The local interpolation error can then be estimated by transforming to the reference element, bounding the error there, and transforming back (a so-called **scaling argument**).

Local interpolation error

Theorem (Local interpolation error)

Let $(\hat{K}, \hat{\mathcal{P}}, \hat{\mathcal{N}})$ *be a finite element with* $P_{k-1} \subset \hat{\mathcal{P}}$ *for some* $k \geq 1$ *and* $\hat{\mathcal{N}}$ *bounded on* $W^{k,p}(\hat{K}), 1 \leq p \leq \infty$. *For any element* $(K, \mathcal{P}, \mathcal{N})$ *affine equivalent to* $(\hat{K}, \hat{\mathcal{P}}, \hat{\mathcal{N}})$ *by the affine transformation* T_K, *there exists a constant* $c > 0$ *independent of* K *such that for any* $v \in W^{k,p}(K)$,

$$|v - \mathcal{I}_K v|_{W^{l,p}(K)} \leq c h_K^{k-l} \sigma_K^l |v|_{W^{k,p}(K)} \quad \text{for all } 0 \leq l \leq k$$

Proof.

Let $\hat{v} := v \circ T_K$. We know $\mathcal{I}_{\hat{K}} \hat{v} = (\mathcal{I}_K v) \circ T_K$ (i.e., interpolating the transformed function is equivalent to transforming the interpolated function). Hence, we can apply Lemma to $(v - \mathcal{I}_K v)$ and use Theorem to obtain (with a generic constant c that can change from line to line)

$$\begin{aligned} |v - \mathcal{I}_K v|_{W^{l,p}(K)} &\leq c \|A_K^{-1}\|^l \, |\det(A_K)|^{\frac{1}{p}} \, |\hat{v} - \mathcal{I}_{\hat{K}} \hat{v}|_{W^{l,p}(\hat{K})} \\ &\leq c \|A_K^{-1}\|^l \, |\det(A_K)|^{\frac{1}{p}} \, |\hat{v}|_{W^{k,p}(\hat{K})} \leq c \|A_K^{-1}\|^l \|A_K\|^k |v|_{W^{k,p}(K)} \\ &\leq c \left(\|A_K^{-1}\| \|A_K\| \right)^l \|A_K\|^{k-l} |v|_{W^{k,p}(K)}. \end{aligned}$$

This follows from the last Lemma and the fact that $h_{\hat{K}}$ and $\rho_{\hat{K}}$ are independent of K. □

Shape regular

To obtain an estimate for the global interpolation error, which should converge to zero as $h \to 0$, we need to have a uniform bound (independent of K and h) of the condition number σ_K. This requires a further assumption on the triangulation.

A triangulation $\mathcal{T}$ is called **shape regular** if there exists a constant κ independent of $h := \max_{K \in \mathcal{T}} h_K$ such that

$$\sigma_K \leq \kappa \quad \text{for all } K \in \mathcal{T}$$

(For triangular elements, e.g., this holds if all interior angles are bounded from below.)

Global interpolation error

Using this upper bound and summing over all elements, we obtain an estimate for the global interpolation error.

Theorem (Global interpolation error)

*Let $\mathcal{T}$ be a **shape regular** affine triangulation of $\Omega \subset \mathbb{R}^n$ with the reference element $(\hat{K}, \hat{\mathcal{P}}, \hat{\mathcal{N}})$ satisfying the requirements of the last Theorem for some $k \geq 1$. Then there exists a constant $c > 0$ independent of h such that for all $v \in W^{k,p}(\Omega)$,*

$$\|v - \mathcal{I}_{\mathcal{T}} v\|_{L^p(\Omega)} + \sum_{l=1}^{k} h^l \left(\sum_{K \in \mathcal{T}} |v - \mathcal{I}_K v|^p_{W^{l,p}(K)} \right)^{\frac{1}{p}} \leq c h^k |v|_{W^{k,p}(\Omega)}, \ 1 \leq p < \infty,$$

$$\|v - \mathcal{I}_{\mathcal{T}} v\|_{L^\infty(\Omega)} + \sum_{l=1}^{k} h^l \max_{K \in \mathcal{T}} |v - \mathcal{I}_K v|_{W^{l,\infty}(K)} \leq c h^k |v|_{W^{k,\infty}(\Omega)}.$$

Similar estimates can be obtained for elements based on the tensor product spaces Q_k^2.

Local inverse estimate

The above theorems estimated the interpolation error in a coarser norm (i.e., $l \le k$) than the given function to be interpolated. In general, the converse (estimating a finer norm by a coarser one) is not possible; however, for the discrete approximations $v_h \in V_h$, such so-called inverse estimates can be established.

Local estimates follow as above from a scaling argument, using the equivalence of norms on the finite dimensional space $\hat{\mathcal{P}}$ in place of the Bramble-Hilbert lemma.

Theorem (local inverse estimate)

Let $(\hat{K}, \hat{\mathcal{P}}, \hat{\mathcal{N}})$ *be a finite element with* $\hat{\mathcal{P}} \subset W^{l,p}(\hat{K})$ *for an* $l \ge 0$ *and* $1 \le p \le \infty$*. For any element* $(K, \mathcal{P}, \mathcal{N})$ *with* $h_K \le 1$ *affine equivalent to* $(\hat{K}, \hat{\mathcal{P}}, \hat{\mathcal{N}})$ *by the affine transformation* T_K*, there exists a constant* $c > 0$ *independent of* K *such that for any* $v_h \in \mathcal{P}$*,*

$$\|v_h\|_{W^{l,p}(K)} \le c h_K^{k-l} \|v_h\|_{W^{k,p}(K)}$$

for all $0 \le k \le l$*.*

Shape-regular vs Quasi-uniform

Why do we need mesh assumptions?

Shape-regular

- Controls shape
- Avoids degenerate elements

Condition

$$\sigma_K = \frac{h_K}{\rho_K} \leq C$$

good bad

Meaning:

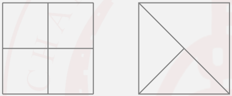
- no very flat / skinny elements

Quasi-uniform

- Controls size
- Ensures uniform scale

Condition

$$h_K \sim h \quad (\forall K)$$



uniform non-uniform

Meaning:

- no very small / very large elements

Global inverse estimate

For uniform global estimates, we need a lower bound on h_K^{-1}. A triangulation $\mathcal{T}$ is called **quasi-uniform** if it is shape regular and there exists a $\tau \in (0,1]$ such that $h_K \geq \tau h$ for all $K \in \mathcal{T}$. By summing over the local estimates, we obtain the following global estimate.

Theorem (global inverse estimate)

*Let $\mathcal{T}$ be a **quasi-uniform** affine triangulation of $\Omega \subset \mathbb{R}^n$ with the reference element $(\hat{K}, \hat{\mathcal{P}}, \hat{\mathcal{N}})$ satisfying the requirements of the last Theorem for an $l \geq 0$. Then there exists a constant $c > 0$ independent of h such that for all $v_h \in V_h := \{v \in L^p(\Omega) : v|_K \in \mathcal{P}, K \in \mathcal{T}\}$*

$$\left(\sum_{K \in \mathcal{T}} \|v_h\|^p_{W^{l,p}(K)} \right)^{\frac{1}{p}} \leq c h^{k-l} \left(\sum_{K \in \mathcal{T}} \|v_h\|^p_{W^{k,p}(K)} \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty,$$

$$\max_{K \in \mathcal{T}} \|v_h\|_{W^{l,\infty}(K)} \leq c h^{k-l} \left(\max_{K \in \mathcal{T}} \|v_h\|_{W^{k,\infty}(K)} \right),$$

for all $0 \leq k \leq l$

C^0 finite element space

We can now give error estimates for the conforming finite element approximation of elliptic boundary value problems using Lagrange elements. Let a reference element $(\hat{K}, \hat{\mathcal{P}}, \hat{\mathcal{N}})$ and a triangulation $\mathcal{T}$ using affine equivalent elements be given. Denoting the affine transformation from the reference element to the element $(K, \mathcal{P}, \mathcal{N})$ by $T_K : \hat{x} \mapsto A_K \hat{x} + b_K$, we can define the corresponding C^0 finite element space by

$$V_h := \left\{ v_h \in C^0(\bar{\Omega}) : (v_h|_K \circ T_K) \in \hat{\mathcal{P}} \text{ for all } K \in \mathcal{T} \right\} \cap V$$

(the intersection being necessary in case of Dirichlet conditions).

Priori Error Estimates

By Céa's lemma, the discretization error is bounded by the best-approximation error, which in turn can be bounded by the interpolation error.

Theorem (H^1 norm)

Let $u \in H^1(\Omega)$ be the solution of the boundary value problem together with appropriate boundary conditions. Let $\mathcal{T}$ be a shape regular affine triangulation of $\Omega \subset \mathbb{R}^n$ with the reference element $(\hat{K}, \hat{\mathcal{P}}, \hat{\mathcal{N}})$ satisfying $P_{k-1} \in \hat{\mathcal{P}}$ for some $k \geq 1$, and let $u_h \in V_h$ be the corresponding Galerkin approximation. If $u \in H^m(\Omega)$ for $\frac{n}{2} < m < k$, then there exists $c > 0$ independent of h and u such that

$$\|u - u_h\|_{H^1(\Omega)} \leq ch^{m-1}|u|_{H^m(\Omega)}$$

Proof.

Since $m > \frac{n}{2}$, the Sobolev embedding Theorem implies that $u \in C^0(\bar{\Omega})$ and hence that the local (pointwise) interpolant is well defined. In addition, the nodal interpolation preserves homogeneous Dirichlet boundary conditions. Hence $\mathcal{I}_{\mathcal{T}} u \in V_h$, and Céa's lemma yields

$$\|u - u_h\|_{H^1(\Omega)} \leq c \inf_{v_h \in V_h} \|u - v_h\|_{H^1(\Omega)} \leq c \|u - \mathcal{I}_{\mathcal{T}} u\|_{H^1(\Omega)}$$

Theorem (global Interpolation error) for $p = 2$, $l = 1$, and $k = m$ further implies

$$\|u - \mathcal{I}_{\mathcal{T}} u\|_{H^1(\Omega)} \leq ch^{m-1} |u|_{H^m(\Omega)}$$

and the claim follows by combining these estimates. □

Priori Error Estimates

Theorem (L^2 norm)

Under the assumptions of the last Theorem, there exists $c > 0$ such that

$$\|u - u_h\|_{L^2(\Omega)} \leq ch^m |u|_{H^m(\Omega)}$$

Proof.

By the Sobolev embedding Theorem, the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is continuous. Thus, the Aubin-Nitsche lemma yields

$$\|u - u_h\|_{L^2(\Omega)} \leq c \, \|u - u_h\|_{H^1(\Omega)} \sup_{g \in L^2(\Omega)} \left(\frac{1}{\|g\|_{L^2(\Omega)}} \inf_{v_h \in V_h} \|\varphi_g - v_h\|_{H^1(\Omega)} \right),$$

where φ_g is the solution of the adjoint problem with right-hand side g. Estimating the best approximation in V_h by the interpolant and using Theorem (global Interpolation error), we obtain

$$\inf_{v_h \in V_h} \|\varphi_g - v_h\|_{H^1(\Omega)} \leq \|\varphi_g - \mathcal{I}_{\mathcal{T}} \varphi_g\|_{H^1(\Omega)} \leq ch \, |\varphi_g|_{H^2(\Omega)} \leq ch \|g\|_{L^2(\Omega)}$$

by the well-posedness of the adjoint problem. Combining this inequality with the one from the last Theorem yields the claimed estimate. □