有限元方法

Finite Element Methods

Chapter 4: Finite Element Space

主讲人: 李琦

liqihao@chd.edu.cn

School of Science, Chang'an University

Outline

FINITE ELEMENT SPACE

Finite element methods are a special case of Galerkin methods, where the finite-dimensional subspace consists of **piecewise polynomials**. To construct these subspaces, we proceed in two steps:

1. We define a **reference element** and study **polynomial interpolation** on this element.

2. We use suitably transformed copies of the reference element to partition the given domain and discuss how to **construct a global interpolant from local interpolants** on each element.

We then follow the same steps in proving interpolation error estimates for functions in Sobolev spaces.

Finite Element

We follow Ciarlet's definition of a finite element (Ciarlet 1978).

Definition

A **finite element** is a triple $(K, \mathcal{P}, \mathcal{N})$ where

(i) $K \subset \mathbb{R}^n$ is a simply connected bounded open set with piecewise smooth boundary (the **element domain**, or simply element if there is no possibility of confusion);

(ii) $\mathcal{P}$ is a finite-dimensional space of functions defined on K (the space of **shape functions**);

(iii) $\mathcal{N} = \{N_1, \ldots, N_d\}$ is a basis of $\mathcal{P}'$ (the set of **nodal variables** or **degrees of freedom**).

Here $\mathcal{P}'$ denotes the algebraic dual of $\mathcal{P}$, i.e., the space of linear functionals on $\mathcal{P}$. As we will see, condition (iii) guarantees that the interpolation problem on K using functions in $\mathcal{P}$ - and hence the Galerkin approximation - is well-posed. The nodal variables will play the role of interpolation conditions.

Definition

Let $(K, \mathcal{P}, \mathcal{N})$ be a finite element. A basis $\{\psi_1, \ldots, \psi_d\}$ of $\mathcal{P}$ is called **dual basis** or **nodal basis** to $\mathcal{N}$ if $N_i(\psi_j) = \delta_{ij}$.

Example

For the linear finite elements in one dimension, $K = (0, 1), \mathcal{P} = P_1$ is the space of linear polynomials, and $\mathcal{N} = \{N_1, N_2\}$ are the point evaluations $\mathcal{N}_1(v) = v(0)$, $\mathcal{N}_2(v) = v(1)$ for every $v \in \mathcal{P}$. The nodal basis is given by $\psi_1(x) = 1 - x$ and $\psi_2(x) = x$.

Condition 3 is the only one that is difficult to verify. The following lemma simplifies this task.

Lemma

Let $\mathcal{P}$ be a d-dimensional vector space and let $\{N_1, \ldots, N_d\}$ be a subset of $\mathcal{P}'$. Then the following statements are equivalent:

(a) $\{N_1, \ldots, N_d\}$ *is a basis of $\mathcal{P}'$;*

(b) *if $v \in \mathcal{P}$ satisfies $N_i(v) = 0$ for all $1 \leq i \leq d$, then $v = 0$.*

Proof.

Let $\{\phi_1, \ldots, \phi_d\}$ be some basis for $\mathcal{P}$. $\{N_1, \ldots, N_d\}$ is a basis for $\mathcal{P}'$ if and only if for any $L \in \mathcal{P}'$, there exist (unique) $\alpha_i, 1 \le i \le d$, such that

$$L = \alpha_1 N_1 + \ldots + \alpha_d N_d$$

(because $d = \dim \mathcal{P} = \dim \mathcal{P}'$). Using the basis of $\mathcal{P}$,

$$y_i := L(\phi_i) = \alpha_1 N_1(\phi_i) + \ldots + \alpha_d N_d(\phi_i), \quad i = 1, \ldots, d$$

Let $\mathbf{B} = (N_j(\phi_i)), i, j = 1, \ldots, d$. Thus, (a) is equivalent to $\mathbf{B}\alpha = y$ is always solvable, which is the same as $\mathbf{B}$ being invertible.

On the other hand, given any $v \in \mathcal{P}$, we can write $v = \beta_1 \phi_1 + \ldots + \beta_d \phi_d$. $N_i v = 0$ means that $\beta_1 N_i(\phi_1) + \ldots + \beta_d N_i(\phi_d) = 0$. Therefore, (b) is equivalent to

$$\beta_1 N_i(\phi_1) + \ldots + \beta_d N_i(\phi_d) = 0 \quad \text{for } i = 1, \ldots, d$$

$$\Longrightarrow \beta_1 = \ldots = \beta_d = 0.$$

Let $\mathbf{C} = (N_i(\phi_j)), i, j = 1, \ldots, d$. Then (b) is equivalent to $\mathbf{C}x = 0$ only has trivial solutions, which is the same as $\mathbf{C}$ being invertible. But $\mathbf{C} = \mathbf{B}^T$. Therefore, (a) is equivalent to (b). □

Note that (b) in particular implies that the interpolation problem using functions in $\mathcal{P}$ with interpolation conditions $\mathcal{N}$ is uniquely solvable.

Definition

We say that $\mathcal{N}$ **determines** $\mathcal{P}$ if $\psi \in \mathcal{P}$ with $N(\psi) = 0 \ \ \forall N \in \mathcal{N}$ implies that $\psi = 0$.

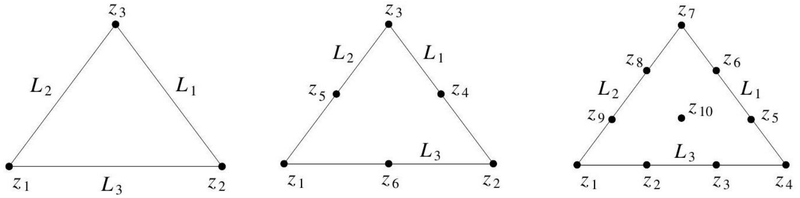
To construct a finite element, one usually proceeds in the following way:

1. choose an element domain K (e.g., a triangle),
2. choose a polynomial space $\mathcal{P}$ of a given degree k (e.g., linear functions),
3. choose d degrees of freedom $\mathcal{N} = \{N_1, \ldots, N_d\}$, where d is the dimension of $\mathcal{P}$, such that the corresponding interpolation problem has a unique solution,
4. compute the nodal basis of $\mathcal{P}$ with respect to $\mathcal{N}$.

The last step amounts to solving for $1 \leq j \leq d$ the concrete interpolation problems $N_i(\psi_j) = \delta_{ij}$.

Lemma (Polynomial division)

Let $L \neq 0$ be a linear-affine functional on $\mathbb{R}^n$ and P be a polynomial of total degree $d \geq 1$ with $P(x) = 0$ for all x with $L(x) = 0$. Then there exists a polynomial Q of total degree $d - 1$ such that $P = LQ$.



(a) linear Lagrage element (b) quadratic Lagrange element (c) cubic Lagrange element

Figure: Triangular finite elements. Filled circles denote point evaluation, open circles gradient evaluations.

Triangular elements Let K be a triangle and $P_k = \left\{\sum_{|\alpha|\leq k} c_\alpha x^\alpha : c_\alpha \in \mathbb{R}\right\}$ denote the space of all bivariate polynomials of total degree less than or equal k, e.g., $P_2 = \operatorname{span}\left\{1, x_1, x_2, x_1^2, x_2^2, x_1 x_2\right\}$. It is straightforward to verify that P_k (and hence P_k') is a vector space of dimension $\frac{1}{2}(k+1)(k+2)$. Two types of interpolation conditions: **function values** (Lagrange interpolation) and **gradient values** (Hermite interpolation).

Linear Lagrange elements

Let $k=1$ and take $\mathcal{P}=P_1$ (hence the dimension of $\mathcal{P}$ and $\mathcal{P}'$ is 3) and $\mathcal{N}=\{N_1, N_2, N_3\}$ with $N_i(v)=v\left(z_i\right)$, where z_1, z_2, z_3 are the vertices of K. We need to show that condition (iii) holds.

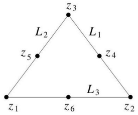
Suppose that $v \in P_1$ satisfies $v\left(z_1\right)=v\left(z_2\right)=v\left(z_3\right)=0$. Since v is linear, it must also vanish on each line connecting the vertices, which can be defined as the zero-sets of the (non-constant) linear functions L_1, L_2, L_3. Hence, by Lemma **Polynomial division**, there exists a constant (i.e., polynomial of degree 0) c such that, e.g., $v=cL_1$. Now let z_1 be the vertex not on the edge defined by L_1. Then

$$0=v\left(z_1\right)=cL_1\left(z_1\right)$$

Since $L_1\left(z_1\right) \neq 0$ (otherwise the linear functional L_1 would be identically zero), this implies $c=0$ and thus $v=0$.

Let $k = 2$ and take $\mathcal{P} = \mathcal{P}_2$ and $\mathcal{N}_2 = \{N_1, N_2, \ldots, N_6\}$ $(\dim \mathcal{P}_2 =$ 6) where

$$N_i(v) = \begin{cases} v\left(i^{th} \text{ vertex }\right), & i = 1,2,3, \\ v(\text{ midpoint of the } (i-3) \text{ edge }), & \\ (\text{ or any other point on the } i-3 \text{ edge }) & i = 4,5,6. \end{cases}$$



We need to check that $\mathcal{N}_2$ determines $\mathcal{P}_2$. Let L_1, L_2 and L_3 be non-trivial linear functions that define the edges of the triangle.

Suppose that $P \in \mathcal{P}_2$ vanishes at $z_1, z_2, \ldots, z_6$. Since $\left.P\right|_{L_1}$ is a quadratic function of one variable that vanishes at three points, $P = 0$ on L_1. By Lemma **Polynomial division** we can write $P = L_1 Q_1$ where $\deg Q_1 = (\deg P) - 1 = 2 - 1 = 1$. But P also vanishes on L_2. Therefore, $\left.L_1 Q_1\right|_{L_2} = 0$. Hence, on L_2, either $L_1 = 0$ or $Q_1 = 0$. But L_1 can equal zero only at one point of L_2 since we have a non-degenerate triangle. Therefore, $Q_1 = 0$ on L_2, except possibly at one point. By continuity, we have $Q_1 \equiv 0$ on L_2.

By Lemma **Polynomial division**, we can write $Q_1 = L_2 Q_2$, where $\deg Q_2 = (\deg L_2) - 1 = 1 - 1 = 0$. Hence, Q_2 is a constant (say c), and we can write $P = c L_1 L_2$. But $P(z_6) = 0$ and z_6 does not lie on either L_1 or L_2. Therefore,

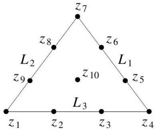
$$0 = P(z_6) = c L_1(z_6) L_2(z_6) \quad \Longrightarrow \quad c = 0$$

since $L_1(z_6) \neq 0$ and $L_2(z_6) \neq 0$. Thus, $P \equiv 0$.

Cubic Lagrange elements

Let $k=3$ and take $\mathcal{P}=\mathcal{P}_3$. Let $\mathcal{N}_3=\{N_i : i=1,2,\ldots,10(=\dim\mathcal{P}_3)\}$ where

$N_i(v)=v\left(z_i\right), \quad i=1,2,\ldots,9\,(z_i$ distinct points on edges)
and
$N_{10}(v)=v($ any interior point $)$.



We show that $\mathcal{N}_3$ determines $\mathcal{P}_3$. Let L_1, L_2 and L_3 be non-trivial linear functions that define the edges of the triangle.

Suppose that $P\in\mathcal{P}_3$ vanishes at z_i for $i=1,2,\ldots,10$. Applying Lemma **Polynomial division** three times along with the fact that $P\left(z_i\right)=0$ for $i=1,2,\ldots,9$, we can write $P=cL_1L_2L_3$. But

$$0=P\left(z_{10}\right)=cL_1\left(z_{10}\right)L_2\left(z_{10}\right)L_3\left(z_{10}\right) \quad\Longrightarrow\quad c=0$$

since $L_i\left(z_{10}\right)\neq 0$ for $i=1,2,3$. Thus, $P\equiv 0$.

In general for $k \geq 1$, we let $\mathcal{P} = \mathcal{P}_k$. For $\mathcal{N}_k = \{N_i : i = 1, 2, \ldots, \frac{1}{2}(k+1)(k+2)\}$, we choose evaluation points at

3 vertex nodes,

$3(k-1)$ distinct edge nodes and

$\frac{1}{2}(k-2)(k-1)$ interior points.

(The interior points are chosen, by induction, to determine $\mathcal{P}_{k-3}$.) Note that these choices suffice since

$$3 + 3(k-1) + \frac{1}{2}(k-2)(k-1)$$
$$= 3k + \frac{1}{2}\left(k^2 - 3k + 2\right) = \frac{1}{2}\left(k^2 + 3k + 2\right) = \frac{1}{2}(k+1)(k+2) = \dim \mathcal{P}_k$$

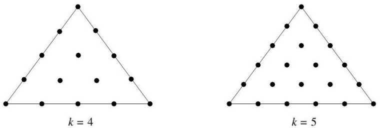
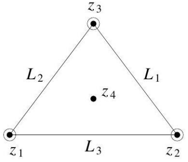


Figure: quartic and quintic Lagrange triangles.

Cubic Hermite elements

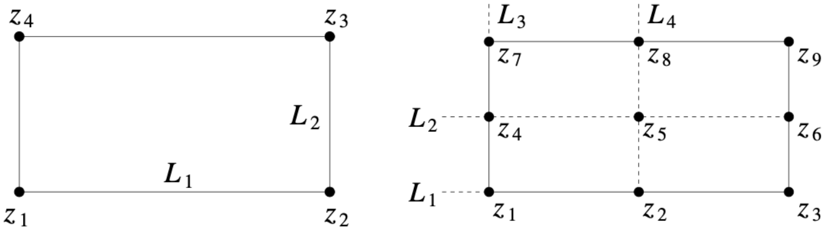
Let $\mathcal{P} = \mathcal{P}_3$. Let "$\bullet$" denote evaluation at the point and " $\bigcirc$ " denote evaluation of the gradient at the center of the circle. Note that the latter corresponds to two distinct nodal variables, but the particular representation of the gradient is not unique. We claim that $\mathcal{N} = \{N_1, N_2, \ldots, N_{10}\}$ determines $\mathcal{P}_3\,(\dim \mathcal{P}_3 = 10)$.



Let L_1, L_2 and L_3 again be non-trivial linear functions that define the edges of the triangle. Suppose that for a polynomial $P \in \mathcal{P}_3, N_i(P) = 0$ for $i = 1, 2, \ldots, 10$. Restricting P to L_1, we see that z_2 and z_3 are double roots of P since $P(z_2) = 0, P'(z_2) = 0$ and $P(z_3) = 0, P'(z_3) = 0$, where $'$ denotes differentiation along the straight line L_1. But the only third order polynomial in one variable with four roots is the zero polynomial, hence $P \equiv 0$ along L_1. Similarly, $P \equiv 0$ along L_2 and L_3. We can, therefore, write $P = cL_1L_2L_3$. But

$$0 = P(z_4) = cL_1(z_4)\,L_2(z_4)\,L_3(z_4) \quad \Longrightarrow \quad c = 0$$

because $L_i(z_4) \neq 0$ for $i = 1, 2, 3$.



(a) bilinear Lagrange element (b) biquadratic Lagrange element

Figure: Rectangular finite elements. Filled circles denote point evaluation.

Rectangular elements For rectangular elements, we consider the vector space (**tensor-product**)

$$Q_k = \left\{ \sum_j c_j p_j\left(x_1\right) q_j\left(x_2\right) : c_j \in \mathbb{R}, p_j, q_j \in P_k \right\}$$

of products of univariate polynomials of degree up to k, which has dimension $(k+1)^2$(e.g., $Q_2 = \operatorname{span}\left\{1, x_1, x_2, x_1 x_2, x_1^2 x_2, x_1 x_2^2, x_1^2, x_2^2\right\}$).

Bilinear Lagrange elements

Let K be any rectangle, $\mathcal{P} = \mathcal{Q}_1$ (hence the dimension of $\mathcal{P}$ and $\mathcal{P}'$ is 4) and $\mathcal{N} = \{N_1, N_2, N_3, N_4\}$ with $N_i(v) = v(z_i)$, where z_1, z_2, z_3, z_4 are the vertices of K.

Suppose that the polynomial $P \in \mathcal{Q}_1$ vanishes at z_1, z_2, z_3 and z_4. The restriction of P to any side of the rectangle is a first-order polynomial of one variable. Therefore, we can write $P = cL_1L_2$ for some constant c. But

$$0 = P(z_4) = cL_1(z_4)L_2(z_4) \quad \Longrightarrow \quad c = 0$$

since $L_1(z_4) \neq 0$ and $L_2(z_4) \neq 0$. Thus, $P \equiv 0$.

Biquadratic Lagrange elements

Let K be any rectangle, $\mathcal{P} = \mathcal{Q}_2$ (hence the dimension of $\mathcal{P}$ and $\mathcal{P}'$ is 9) , and $\mathcal{N} = \{N_1, \ldots, N_9\}$ with $N_i(v) = v(z_i)$, where z_1, z_2, z_3, z_4 are the vertices of K, z_5, z_6, z_7, z_8 are the edge midpoints and z_9 is the centroid of K.

Suppose that a polynomial $P \in \mathcal{Q}_2$ vanishes at z_i, for $i = 1, \ldots, 9$. Then we can write $P = cL_1L_2L_3L_4$ for some constant c. But

$$0 = P(z_9) = cL_1(z_9)\,L_2(z_9)\,L_3(z_9)\,L_4(z_9) \quad \Longrightarrow \quad c = 0$$

since $L_i(z_9) \neq 0$ for $i = 1, 2, 3, 4$.

Interpolant

We wish to estimate the error of the best approximation of a function in a finite element space. An upper bound for this approximation is given by stitching together interpolating polynomials on each element.

Definition (Local interpolant)

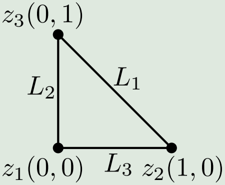
Let $(K, \mathcal{P}, \mathcal{N})$ be a finite element and let $\{\psi_1, \ldots, \psi_d\}$ be the corresponding nodal basis of $\mathcal{P}$. For a given function v such that $N_i(v)$ is defined for all $1 \leq i \leq d$, the **local interpolant** of v is defined as

$$\mathcal{I}_K v = \sum_{i=1}^{d} N_i(v)\psi_i$$

The local interpolant can be explicitly constructed once the nodal basis is known. This can be simplified significantly if the reference element domain is chosen as, e.g., the unit simplex.

Example

Let K be the triangle depicted in Figure, $\mathcal{P} = \mathcal{P}_1$, $\mathcal{N} = \{N_1, N_2, N_3\}$, and $f = e^{xy}$. We want to find $\mathcal{I}_K f$.



By definition, $\mathcal{I}_K f = N_1(f)\phi_1 + N_2(f)\phi_2 + N_3(f)\phi_3$. We must therefore determine ϕ_1, ϕ_2 and ϕ_3. The line L_1 is given by $y = 1 - x$. We can write $\phi_1 = cL_1 = c(1 - x - y)$. But $N_1(\phi_1) = 1$ implies that $c = \phi_1(z_1) = 1$, hence $\phi_1 = 1 - x - y$. Similarly, $\phi_2 = x$ and $\phi_3 = y$. Therefore,

$$\begin{aligned} \mathcal{I}_K f &= N_1(f)(1 - x - y) + N_2(f)x + N_3(f)y \\ &= 1 - x - y + x + y \quad (\text{ since } f = e^{xy}) \\ &= 1 \end{aligned}$$

Properties of the local interpolant

Lemma

Let $(K, \mathcal{P}, \mathcal{N})$ *be a finite element and* $\mathcal{I}_K$ *the local interpolant. Then*

1. *the mapping* $v \mapsto \mathcal{I}_K v$ *is linear;*
2. $N_i(\mathcal{I}_K v) = N_i(v), 1 \leq i \leq d$*;*
3. $\mathcal{I}_K(v) = v$ *for all* $v \in \mathcal{P}$*. In particular,* $\mathcal{I}_K$ *is a projection (idempotent), i.e.,* $\mathcal{I}_K^2 = \mathcal{I}_K$.

Proof.

The claim (i) follows directly from the linearity of the N_i. For (ii), we use the definition of $\mathcal{I}_K$ and ψ_i to obtain

$$N_i(\mathcal{I}_K v) = N_i\left(\sum_{j=1}^{d} N_j(v)\psi_j\right) = \sum_{j=1}^{d} N_j(v) N_i(\psi_j) = \sum_{j=1}^{d} N_j(v)\delta_{ij} = N_i(v)$$

for all $1 \leq i \leq d$ and arbitrary v. This implies that $N_i(v - \mathcal{I}_K v) = 0$ for all $1 \leq i \leq d$, and hence by Lemma **Polynomial division** that $\mathcal{I}_K v = v$ and $\mathcal{I}_K^2 f = \mathcal{I}_K(\mathcal{I}_K f) = \mathcal{I}_K f$ since $\mathcal{I}_K f \in \mathcal{P}$. Therefore (iii) holds. □

Global interpolant

We now use the local interpolant on each element to define a global interpolant on a union of elements.

Definition (Subdivision)

A **subdivision** of a bounded open set $\Omega \subset \mathbb{R}^n$ is a finite collection $\mathcal{T}$ of open sets K_i such that

1. $K_i \cap K_j = \emptyset$ if $i \neq j$;
2. $\bigcup_i \bar{K}_i = \bar{\Omega}$.

Definition (Global interpolant)

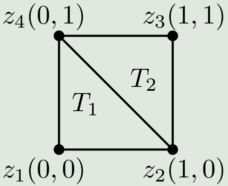
Let $\mathcal{T}$ be a subdivision of Ω such that for each K_i there is a finite element $(K_i, \mathcal{P}_i, \mathcal{N}_i)$ with local interpolant $\mathcal{I}_{K_i}$, and let m be the order of the highest partial derivative appearing in any nodal variable. Then the **global interpolant** $\mathcal{I}_{\mathcal{T}} v$ of $v \in C^m(\bar{\Omega})$ on $\mathcal{T}$ is defined by

$$(\mathcal{I}_{\mathcal{T}} v)\big|_{K_i} = \mathcal{I}_{K_i} v \quad \text{for all } K_i \in \mathcal{T}.$$

Example

Let Ω be the square. The triangulation $\mathcal{T}$ consists of the two triangles T_1 and T_2, as indicated. The finite element on each triangle is the Lagrange element. The dual basis on T_1 is $\{1 - x - y, x, y\}$ and the dual basis on T_2 is $\{1 - x, 1 - y, x + y - 1\}$. Let $f = \sin(\pi(x + y)/2)$. Then

$$\mathcal{I}_{\mathcal{T}} f = \begin{cases} x + y & \text{on } T_1 \\ 2 - x - y & \text{on } T_2 \end{cases}$$



Triangulation

To obtain some regularity of the global interpolant, we need additional assumptions on the subdivision. Roughly speaking, where two elements meet, the corresponding nodal variables have to match as well. For triangular elements, this can be expressed concisely.

Definition (Triangulation)

A triangulation of a bounded open set $\Omega \subset \mathbb{R}^2$ is a subdivision $\mathcal{T}$ of Ω such that

1. every $K_i \in \mathcal{T}$ is a triangle;
2. no vertex of any triangle lies on an edge of another triangle (i.e., no hanging nodes).

Similar conditions can be given for $n \geq 3$ (tetrahedra, simplices), in which case one usually also speaks of triangulations. Note that this supposes that Ω is polyhedral itself. (For non-polyhedral domains, it is possible to use curved elements near the boundary.)

C^m finite element space

Definition (Continuity)

A global interpolant $\mathcal{I}_\mathcal{T}$ has **continuity order** r (in short, "is C^r") if $\mathcal{I}_\mathcal{T} v \in C^r(\bar{\Omega})$ for all $v \in C^m(\bar{\Omega})$ (for which the interpolation is well-defined). In this case, the space

$$V_\mathcal{T} = \left\{ \mathcal{I}_\mathcal{T} v : v \in C^m(\bar{\Omega}) \right\}$$

is called a C^r finite element space.

In particular, to obtain global continuity of the interpolant, we need to make sure that the local interpolants coincide where two element domains meet. This requires that the corresponding nodal variables are compatible.

C^0 finite element space

Theorem

The triangular Lagrange and Hermite elements of fixed degree are all C^0 elements (i.e., lead to C^0 finite element space). More precisely, given a triangulation $\mathcal{T}$ of Ω, it is possible to choose edge nodes for the corresponding elements $(K_i, \mathcal{P}_i, \mathcal{N}_i)\,, K_i \in \mathcal{T}$, such that $\mathcal{I}_{\mathcal{T}} v \in C^0(\bar{\Omega})$ for all $v \in C^m(\bar{\Omega})$, where $m = 0$ for Lagrange and $m = 1$ for Hermite elements.

In order to obtain global interpolation error estimates, we need uniform bounds on the local interpolation errors. For this, we need to be able to compare the local interpolation operators on different elements. This can be done with the following notion of equivalence of elements.

Definition

Let $(\hat{K}, \hat{\mathcal{P}}, \hat{\mathcal{N}})$ be a finite element and $T : \mathbb{R}^n \to \mathbb{R}^n$ be an affine transformation, i.e., $T : \hat{x} \mapsto A\hat{x} + b$ for $A \in \mathbb{R}^{n\times n}$ invertible and $b \in \mathbb{R}^n$. The finite element $(K, \mathcal{P}, \mathcal{N})$ is called **affine equivalent** to $(\hat{K}, \hat{\mathcal{P}}, \hat{\mathcal{N}})$ if

1. $K = \{A\hat{x} + b : \hat{x} \in \hat{K}\}$
2. $\mathcal{P} = \left\{\hat{p} \circ T^{-1} : \hat{p} \in \hat{\mathcal{P}}\right\}$
3. $\mathcal{N} = \left\{N_i : N_i(p) = \hat{N}_i(p \circ T) \text{ for all } p \in \mathcal{P}\right\}$.

A triangulation $\mathcal{T}$ consisting of affine equivalent elements is also called **affine**.

- It is a straightforward exercise to show that the nodal bases of $\hat{\mathcal{P}}$ and $\mathcal{P}$ are related by $\hat{\psi}_i = \psi_i \circ T$. Hence, if the nodal variables on edges are placed symmetrically, triangular Lagrange elements of the same order are affine equivalent, as are triangular Hermite elements.
- The same holds true for rectangular elements. Non-affine equivalent elements (such as isoparametric elements) are useful in treating elements with curved boundaries (for non-polyhedral domains)

Interpolation equivalent

The advantage of this construction is that affine equivalent elements are also **interpolation equivalent** in the following sense.

Lemma

Lemma 4.13. Let $(\hat{K}, \hat{\mathcal{P}}, \hat{\mathcal{N}})$ and $(K, \mathcal{P}, \mathcal{N})$ be two affine equivalent finite elements related by the transformation T_K. Then,

$$\mathcal{I}_{\hat{K}}(v \circ T_K) = (\mathcal{I}_K v) \circ T_K$$

Proof.

Let $\hat{\psi}_i$ and ψ_i be the nodal basis of $\hat{\mathcal{P}}$ and $\mathcal{P}$, respectively. By definition,

$$\mathcal{I}_{\hat{K}}(v \circ T_K) = \sum_{i=1}^{d} \hat{N}_i(v \circ T_K)\,\hat{\psi}_i = \sum_{i=1}^{d} N_i(v)\,(\psi_i \circ T_K) = (\mathcal{I}_K v) \circ T_K.$$

Given a reference element $(\hat{K}, \hat{\mathcal{P}}, \hat{\mathcal{N}})$, we can thus generate a triangulation $\mathcal{T}$ using affine equivalent elements. □