有限元方法

Finite Element Methods

Chapter 3: Variational Formulation of Elliptic BVPs

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Outline

1. VARIATIONAL FORMULATION OF ELLIPTIC BVPS
   - Inner-Product Spaces
   - Hilbert Spaces
   - Projections onto Subspaces
   - Riesz Representation Theorem
   - Formulation of Symmetric Variational Problems
   - Formulation of Nonsymmetric Variational Problems
   - The Lax-Milgram Theorem
   - Estimates for General Finite Element Approximation

Inner-product space

Definition

A **bilinear form**, $b(\cdot,\cdot)$, on a linear space V is a mapping $b : V \times V \longrightarrow \mathbb{R}$ such that each of the maps $v \mapsto b(v,w)$ and $w \mapsto b(v,w)$ is a linear form on V. It is **symmetric** if $b(v,w) = b(w,v)$ for all $v,w \in V$. A (real) **inner product**, denoted by $(\cdot,\cdot)$, is a symmetric bilinear form on a linear space V that satisfies

1. $(v,v) \geq 0 \; \forall v \in V$ and
2. $(v,v) = 0 \Longleftrightarrow v = 0$.

Definition

A linear space V together with an inner product defined on it is called an **inner-product** space and is denoted by $(V,(\cdot,\cdot))$.

- $V = \mathbb{R}^n$, $(x,y) = \sum_{i=1}^{n} x_i y_i$;
- $V = L^2(\Omega),\ \Omega \subseteq \mathbb{R}^n$, $(u,v)_{L^2(\Omega)} = \int_\Omega u(x)v(x)\,dx$;
- $V = W^{k,2}(\Omega),\ \Omega \subseteq \mathbb{R}^n$, $(u,v)_k = \sum_{|\alpha|\leq k} (D^\alpha u, D^\alpha v)_{L^2(\Omega)}$;

Theorem (Schwarz Inequality)

If $(V,(\cdot,\cdot))$ *is an inner-product space, then*

$$|(u,v)| \le (u,u)^{1/2}(v,v)^{1/2}.$$

The equality holds if and only if u and v are linearly dependent.

Proposition

$\|v\| := \sqrt{(v,v)}$ *defines a* ***norm*** *in the inner-product space* $(V,(\cdot,\cdot))$.

Definition (Hilbert space)

Let $(V, (\cdot, \cdot))$ be an inner-product space. If the associated normed linear space $(V, \|\cdot\|)$ is complete, then $(V, (\cdot, \cdot))$ is called a **Hilbert space**.

Definition (subspace)

Let H be a Hilbert space and $S \subset H$ be a linear subset that is closed in H. (Recall that S linear means that $u, v \in S, \alpha \in \mathbb{R} \Longrightarrow u + \alpha v \in S$.) Then S is called a **subspace of** H.

Proposition

If S is a subspace of H, then $(S, (\cdot, \cdot))$ is also a Hilbert space.

Proof.

Proof. $(S, \|\cdot\|)$ is complete because S is closed in H under the norm $\|\cdot\|$. □

Examples of subspaces of Hilbert spaces

1. H and $\{0\}$ are the obvious extreme cases.
2. Let $T : H \longrightarrow K$ be a continuous linear map of H into another linear space. Then $\ker T$ is a subspace.
3. Let $x \in H$ and define $x^\perp := \{v \in H : (v, x) = 0\}$. Then $x^\perp$ is a subspace of H. To see this, note that $x^\perp = \ker L_x$, where L_x is the linear functional

$$L_x : v \mapsto (v, x).$$

By the Schwarz inequality,

$$|L_x(v)| \leq \|x\|\|v\|$$

implying that L_x is bounded and therefore continuous. This proves that $x^\perp$ is a subspace of H in view of the previous example.

4. Let $M \subset H$ be a subset and define $M^\perp := \{v \in H : (x, v) = 0 \ \forall x \in M\}$. Note that

$$M^\perp = \bigcap_{x \in M} x^\perp$$

and each $x^\perp$ is a (closed) subspace of H. Thus, $M^\perp$ is a subspace of H.

Proposition

Let H be a Hilbert space.

1. *For any subsets $M, N \subset H, M \subset N \Longrightarrow N^\perp \subset M^\perp$.*
2. *For any subset M of H containing zero, $M \cap M^\perp = \{0\}$.*
3. $\{0\}^\perp = H$.
4. $H^\perp = \{0\}$.

Theorem (Parallelogram Law)

Let $\|\cdot\|$ be the norm associated with the inner product $(\cdot,\cdot)$ on H. We have

$$\|v+w\|^2 + \|v-w\|^2 = 2\left(\|v\|^2 + \|w\|^2\right).$$

Projections onto Subspaces

An essential geometric fact about Hilbert spaces.

Proposition

Let M be a subspace of the Hilbert space H. Let $v \in H \backslash M$ and define $\delta := \inf\{\|v - w\| : w \in M\}$. (Note that $\delta > 0$ since M is closed in H.) Then there exists $w_0 \in M$ such that

1. $\|v - w_0\| = \delta$, *i.e., there exists a closest point $w_0 \in M$ to v, and*
2. $v - w_0 \in M^\perp$.

Given a subspace M of H and $v \in H$, we can write $v = w_0 + w_1$, where $w_0 \in M$ and $w_1 \, (= v - w_0) \in M^\perp$.

- This decomposition of an element $v \in H$ is unique. In fact, from

$$w_0 + w_1 = v = z_0 + z_1, \quad w_0, z_0 \in M, \quad w_1, z_1 \in M^\perp$$

we obtain $M \ni w_0 - z_0 = -(w_1 - z_1) \in M^\perp$. Since $M \cap M^\perp = \{0\}, w_0 = z_0$ and $w_1 = z_1$. This shows that the decomposition is unique.

- Therefore, we can define the following operators

$$P_M : H \longrightarrow M, \quad P_M^{\perp} : H \longrightarrow M^{\perp}$$

where the respective definitions of P_M and $P_M^{\perp}$ are given by

$$P_M v = \begin{cases} v & \text{if } v \in M, \\ w_0 & \text{if } v \in H \backslash M; \end{cases} \qquad P_M^{\perp} v = \begin{cases} 0 & \text{if } v \in M, \\ v - w_0 & \text{if } v \in H \backslash M. \end{cases}$$

Proposition

Given a subspace M of H and $v \in H$, there is a unique decomposition

$$v = P_M v + P_{M^{\perp}} v$$

where $P_M : H \longrightarrow M$ and $P_{M^{\perp}} : H \longrightarrow M^{\perp}$. In other words,

$$H = M \oplus M^{\perp}.$$

The operators P_M and $P_{M^{\perp}}$ defined above are linear operators.

Definition

An operator P on a linear space V is a projection if $P^2 = P$ (**idempotent property**), i.e., $Pz = z$ for all z in the image of P.

Remark

The fact that P_M is a projection follows from its definition. That $P_M^\perp$ is also follows from the observation that $P_M^\perp = P_{M^\perp}$.

Riesz Representation Theorem

Given $u \in H$, recall that a continuous linear functional L_u can be defined on H by

$$L_u(v) = (u, v).$$

The following theorem proves that the converse is also true.

Theorem (Riesz Representation Theorem)

Any continuous linear functional L on a Hilbert space H can be represented uniquely as

$$L(v) = (u, v)$$

for some $u \in H$. Furthermore, we have

$$\|L\|_{H'} = \|u\|_H.$$

Proof.

□

Remark

According to the Riesz Representation Theorem, there is a natural isometry between H and H' $(u \in H \longleftrightarrow L_u \in H')$. For this reason, H and H' are often identified. For example, we can write $W_2^m(\Omega) \cong W_2^{-m}(\Omega)$ (although they are completely different Hilbert spaces). We will use τ to represent the isometry from H' onto H.

Formulation of Symmetric Variational Problems

To get **existence and uniqueness** results for variational formulations of boundary value problems.

Definition

A bilinear form $a(\cdot,\cdot)$ on a normed linear space H is said to be **bounded** (or **continuous**) if $\exists\ C < \infty$ such that

$$|a(v,w)| \leq C\|v\|_H\|w\|_H \quad \forall v, w \in H,$$

and **coercive** on $V \subset H$ if $\exists\ \alpha > 0$ such that

$$a(v,v) \geq \alpha\|v\|_H^2 \quad \forall v \in V.$$

Proposition

Let H be a Hilbert space, and suppose $a(\cdot,\cdot)$ is a symmetric bilinear form that is continuous on H and coercive on a subspace V of H. Then $(V, a(\cdot,\cdot))$ is a Hilbert space.

Proof.

An immediate consequence of the coercivity of $a(\cdot,\cdot)$ is that if $v \in V$ and $a(v,v) = 0$, then $v \equiv 0$. Hence, $a(\cdot,\cdot)$ is an inner product on V.
Now let $\|v\|_E = \sqrt{a(v,v)}$, and suppose that $\{v_n\}$ is a Cauchy sequence in $(V, \|\cdot\|_E)$. By coercivity, $\{v_n\}$ is also Cauchy in $(H, \|\cdot\|_H)$. Since H is complete, $\exists\ v \in H$ such that $v_n \to v$ in the $\|\cdot\|_H$ norm. Since V is closed in $H, v \in V$. Now, $\|v - v_n\|_E \leq \sqrt{c_1}\, \|v - v_n\|_H$ since $a(\cdot,\cdot)$ is bounded. Hence, $\{v_n\} \to v$ in the $\|\cdot\|_E$ norm, so $(V, \|\cdot\|_E)$ is complete. □

In general, a symmetric variational problem is posed as follows. Suppose that the following three conditions are valid:

$$(\boldsymbol{C}_1) \quad \begin{cases} (1) & (H, (\cdot,\cdot)) \text{ is a Hilbert space.} \\ (2) & V \text{ is a (closed) subspace of } H. \\ (3) & a(\cdot,\cdot) \text{ is a bounded, symmetric bilinear form} \\ & \text{that is coercive on } V. \end{cases}$$

Symmetric Variational Problem

Given $F \in V'$, find $u \in V$ such that

$$(\boldsymbol{W}) \qquad a(u, v) = F(v) \quad \forall v \in V.$$

Theorem

Suppose that conditions (1) - (3) of $(\boldsymbol{C}_1)$ *hold. Then there exists a unique* $u \in V$ *solving the symmetric variational problem* $(\boldsymbol{W})$.

Proof.

$a(\cdot,\cdot)$ is an inner product on V and that $(V, a(\cdot,\cdot))$ is a Hilbert space. Apply the Riesz Representation Theorem. □

Ritz-Galerkin Approximation Problem

(Ritz-Galerkin) Approximation Problem

Given a finite-dimensional subspace $V_h \subset V$ and $F \in V'$, find $u_h \in V_h$ such that

$$(\boldsymbol{W}_h) \qquad a(u_h, v) = F(v) \quad \forall v \in V_h.$$

Theorem

Under the conditions (1)-(3) of $(\boldsymbol{C}_1)$, *there exists a unique* u_h *that solves the (Ritz-Galerkin) approximation problem* $(\boldsymbol{W}_h)$.

Proof.

$(V_h, a(\cdot,\cdot))$ is a Hilbert space in its own right, and $F|_{V_h} \in V_h'$. Apply the Riesz Representation Theorem. □

Galerkin Orthogonality

Proposition (Fundamental Galerkin Orthogonality)

Let u and u_h be solutions to ***Symmetric Variational Problem*** $(\boldsymbol{W})$ *and* ***Ritz-Galerkin Approximation Problem*** $(\boldsymbol{W}_h)$ *respectively. Then*

$$a\left(u-u_h, v\right)=0 \quad \forall v \in V_h .$$

Corollary

$\left\|u-u_h\right\|_E=\min_{v \in V_h}\|u-v\|_E$.

Ritz Method

Remark (The Ritz Method)

If a is coercive and symmetric, $u \in V$ satisfies (W) if and only if u is the minimizer of

$$J(v) := \frac{1}{2}a(v,v) - F(v),$$

over all $v \in V$.

Proof.

For any $u, v \in V$ and $t \in \mathbb{R}$

$$J(u+tv) = J(u) + t(a(u,v) - F(v)) + \frac{t^2}{2}a(v,v)$$

due to the bilinearity and symmetry of a. Assume now that u satisfies $a(u,v) - F(v) = 0$ for all $v \in V$. Then setting $t = 1$, we deduce that for all $v \neq 0$,

$$J(u+v) = J(u) + \frac{1}{2}a(v,v) \geq J(u) + \frac{c_1}{2}\|v\|_V^2 > J(u)$$

Hence, u is the unique minimizer of J. Conversely, if u is the (unique) minimizer of J, every directional derivative of J at u must vanish, which implies that

$$0 = \frac{d}{dt}J(u+tv)\Big|_{t=0} = a(u,v) - F(v)$$

for all $v \in V$. □

Formulation of Nonsymmetric Variational Problems

A nonsymmetric variational problem is posed as follows. Suppose that the following five conditions are valid:

$$(\boldsymbol{C}_2) \qquad \begin{cases} (1)\ (H,(\cdot,\cdot)) \text{ is a Hilbert space.} \\ (2)\ V \text{ is a (closed) subspace of } H. \\ (3)\ a(\cdot,\cdot) \text{ is a bilinear form on } V, \text{ not necessarily symmetric.} \\ (4)\ a(\cdot,\cdot) \text{ is continuous (bounded) on } V. \\ (5)\ a(\cdot,\cdot) \text{ is coercive on } V. \end{cases}$$

Nonsymmetric variational problem

Given $F \in V'$, find $u \in V$ such that

$$(\boldsymbol{W}_2) \qquad a(u,v) = F(v) \quad \forall v \in V.$$

(Galerkin) approximation problem

Given a finite-dimensional subspace $V_h \subset V$ and $F \in V'$, find $u_h \in V_h$ such that

$$(\boldsymbol{W}_{h2}) \qquad a(u_h, v) = F(v) \quad \forall v \in V_h$$

Example

An Interesting Example. Consider the boundary value problem

$$-u'' + u' + u = f \quad \text{on } [0,1] \quad u'(0) = u'(1) = 0.$$

One variational formulation for this is: Take

$$V = H^1(0,1)$$

$$a(u,v) = \int_0^1 (u'v' + u'v + uv)\,dx, \quad F(v) = (f,v).$$

Note that $a(\cdot,\cdot)$ is not symmetric because of the $u'v$ term.

Example

To prove $a(\cdot,\cdot)$ is continuous, observe that

$$\begin{aligned} |a(u,v)| &\leq |(u,v)_{H^1}| + \left| \int_0^1 u'v dx \right| \\ &\leq \|u\|_{H^1}\|v\|_{H^1} + \|u'\|_{L^2}\|v\|_{L^2} \qquad \text{(Schwarz' inequality)} \\ &\leq 2\|u\|_{H^1}\|v\|_{H^1} \end{aligned}$$

Therefore, $a(\cdot,\cdot)$ is continuous (take $c_1 = 2$ in the definition).

To prove $a(\cdot,\cdot)$ is coercive, observe that

$$\begin{aligned} a(v,v) &= \int_0^1 \left(v'^2 + v'v + v^2\right) dx \\ &= \frac{1}{2}\int_0^1 (v'+v)^2\, dx + \frac{1}{2}\int_0^1 \left({v'}^2 + v^2\right) dx \\ &\geq \frac{1}{2}\|v\|_{H^1}^2 \end{aligned}$$

Therefore, $a(\cdot,\cdot)$ is coercive (take $c_2 = 1/2$ in the definition).

Lemma (Contraction Mapping Principle)

Given a Banach space V and a mapping $T : V \longrightarrow V$, satisfying

$$\|Tv_1 - Tv_2\| \leq M \|v_1 - v_2\|$$

for all $v_1, v_2 \in V$ and fixed $M, 0 \leq M < 1$, there exists a unique $u \in V$ such that

$$u = Tu,$$

*i.e. the **contraction mapping** T has a unique **fixed point** u.*

Remark

We actually only need that V is a complete metric space in the lemma.

The Lax-Milgram Theorem

To prove the existence and uniqueness of the solution of the (nonsymmetric) variational problem $(\boldsymbol{W}_2)$. The Lax-Milgram Theorem guarantees both existence and uniqueness of the solution.

Theorem (Lax-Milgram)

Given a Hilbert space $(V, (\cdot,\cdot))$*, a continuous, coercive bilinear form* $a(\cdot,\cdot)$ *and a continuous linear functional* $F \in V'$*, there exists a unique* $u \in V$ *such that*

$$a(u,v) = F(v) \quad \forall v \in V.$$

Proof.

For any $u \in V$, define a functional Au by $Au(v) = a(u,v) \quad \forall v \in V$. Au is linear since

$$\begin{aligned} Au\left(\alpha v_1 + \beta v_2\right) &= a\left(u, \alpha v_1 + \beta v_2\right) \\ &= \alpha a\left(u, v_1\right) + \beta a\left(u, v_2\right) \\ &= \alpha Au\left(v_1\right) + \beta Au\left(v_2\right) \quad \forall v_1, v_2 \in V, \alpha, \beta \in \mathbb{R}. \end{aligned}$$

□

Continued proof.

Au is also continuous since, for all $v \in V$,

$$|Au(v)| = |a(u,v)| \leq C\|u\|\|v\|,$$

where C is the constant from the definition of continuity for $a(\cdot,\cdot)$. Therefore,

$$\|Au\|_{V'} = \sup_{v \neq 0} \frac{|Au(v)|}{\|v\|} \leq C\|u\| < \infty.$$

Thus, $Au \in V'$. Similarly, one can show that the mapping $u \to Au$ is a linear map $V \longrightarrow V'$. Here we also showed that the linear mapping $A : V \longrightarrow V'$ is continuous with $\|A\|_{L(V,V')} \leq C$.

Now, by the **Riesz Representation Theorem**, for any $\phi \in V'$ there exists unique $\tau\phi \in V$ such that $\phi(v) = (\tau\phi, v)$ for any $v \in V$. We must find a unique u such that

$$Au(v) = F(v) \quad \forall v \in V.$$

□

Continued proof.

In other words, we want to find a unique u such that

$$Au = F \quad (\text{ in } V'),$$

or

$$\tau Au = \tau F \quad (\text{ in } V),$$

since $\tau : V' \longrightarrow V$ is a one-to-one mapping. We solve this last equation by using **Contraction Mapping Principle**. We want to find $\rho \neq 0$ such that the mapping $T : V \longrightarrow V$ is a contraction mapping, where T is defined by

$$Tv := v - \rho(\tau Av - \tau F) \quad \forall v \in V.$$

If T is a contraction mapping, then by **Contraction Mapping Principle**, there exists a unique $u \in V$ such that

$$Tu = u - \rho(\tau Au - \tau F) = u,$$

that is, $\rho(\tau Au - \tau F) = 0$, or $\tau Au = \tau F$. □

Continued proof.

It remains to show that such a $\rho \neq 0$ exists. For any $v_1, v_2 \in V$, let $v = v_1 - v_2$. Then

$$\begin{aligned}
\|Tv_1 - Tv_2\|^2 &= \|v_1 - v_2 - \rho\,(\tau A v_1 - \tau A v_2)\|^2 \\
&= \|v - \rho(\tau A v)\|^2 \\
&= \|v\|^2 - 2\rho(\tau A v, v) + \rho^2 \|\tau A v\|^2 \quad (\tau, A \text{ are linear}) \\
&= \|v\|^2 - 2\rho A v(v) + \rho^2 A v(\tau A v) \quad (\text{definition of } \tau) \\
&= \|v\|^2 - 2\rho a(v, v) + \rho^2 a(v, \tau A v) \quad (\text{definition of } A) \\
&\leq \|v\|^2 - 2\rho\alpha \|v\|^2 + \rho^2 C \|v\| \|\tau A v\| \\
&\qquad\qquad (\text{coercivity and continuity of } A) \\
&\leq \left(1 - 2\rho\alpha + \rho^2 C^2\right) \|v\|^2 \quad (A \text{ bounded, } \tau \text{ isometric}) \\
&= \left(1 - 2\rho\alpha + \rho^2 C^2\right) \|v_1 - v_2\|^2 \\
&= M^2 \|v_1 - v_2\|^2 .
\end{aligned}$$

Here, α is the constant in the definition of coercivity of $a(\cdot, \cdot)$. □

Continued proof.

Note that $\|\tau A v\| = \|Av\| \le C\|v\|$ was used in the last inequality. We thus need

$$1 - 2\rho\alpha + \rho^2 C^2 < 1 \text{ for some } \rho, \quad \text{i.e.,}$$

$$\rho\left(\rho C^2 - 2\alpha\right) < 0$$

If we choose $\rho \in \left(0, 2\alpha/C^2\right)$ then $M < 1$ and the proof is complete. □

Remark

Note that $\|u\|_V \le (1/\alpha)\|F\|_{V'}$ where α is the coercivity constant.

Corollary

Under conditions $(\boldsymbol{C}_2)$, the variational problem has a unique solution.

Proof.

Conditions (1) and (2) of $(\boldsymbol{C}_2)$ imply that $(V, (\cdot, \cdot))$ is a Hilbert space. Apply the Lax-Milgram Theorem. □

Corollary

Under the conditions $(\boldsymbol{C}_2)$, the Galerkin approximation problem has a unique solution.

Proof.

Since V_h is a (closed) subspace of V, $(\boldsymbol{C}_2)$ holds with V replaced by V_h. Apply the previous corollary. □

Remark

Note that V_h need not be finite-dimensional for approximation problem to be well-posed.

Estimates for General Finite Element Approximation

Let u be the solution to the variational problem and u_h be the solution to the approximation problem. We now want to estimate the error $\|u - u_h\|_V$. We do so by the following theorem.

Theorem (Céa)

Suppose the conditions (C_2) *hold and that* u *solves variational problem. For the finite element variational problem we have*

$$\|u - u_h\|_V \leq \frac{C}{\alpha} \min_{v \in V_h} \|u - v\|_V,$$

where C *is the continuity constant and* α *is the coercivity constant of* $a(\cdot, \cdot)$ *on* V*.*

Proof.

Since $a(u, v) = F(v)$ for all $v \in V$ and $a(u_h, v) = F(v)$ for all $v \in V_h$ we have (by subtracting)

$$a(u - u_h, v) = 0 \quad \forall v \in V_h.$$

□

Continued proof.

For all $v \in V_h$,

$$\begin{aligned} \alpha \|u-u_h\|_V^2 &\le a(u-u_h, u-u_h) && \text{(by coercivity)} \\ &= a(u-u_h, u-v) + a(u-u_h, v-u_h) \\ &= a(u-u_h, u-v) && (\text{since } v-u_h \in V_h) \\ &\le C\|u-u_h\|_V \|u-v\|_V. && \text{(by continuity)} \end{aligned}$$

Hence,

$$\|u-u_h\|_V \le \frac{C}{\alpha}\|u-v\|_V \quad \forall v \in V_h.$$

Therefore,

$$\begin{aligned} \|u-u_h\|_V &\le \frac{C}{\alpha} \inf_{v\in V_h} \|u-v\|_V \\ &= \frac{C}{\alpha} \min_{v\in V_h} \|u-v\|_V \quad (\text{since } V_h \text{ is closed}) \end{aligned}$$

□

Remark

1. Céa's Theorem shows that u_h is quasi-optimal in the sense that the error $\|u - u_h\|_V$ is proportional to the best it can be using the subspace V_h.
2. In the symmetric case, we proved

$$\|u - u_h\|_E = \min_{v \in V_h} \|u - v\|_E.$$

Hence,

$$\begin{aligned} \|u - u_h\|_V &\le \frac{1}{\sqrt{\alpha}} \|u - u_h\|_E \\ &= \frac{1}{\sqrt{\alpha}} \min_{v \in V_h} \|u - v\|_E \\ &\le \sqrt{\frac{C}{\alpha}} \min_{v \in V_h} \|u - v\|_V \\ &\le \frac{C}{\alpha} \min_{v \in V_h} \|u - v\|_V \end{aligned}$$

the result of Céa's Theorem. This is really the remark about the relationship between the two formulations, namely, that one can be derived from the other.