有限元方法

Finite Element Methods

Chapter 2: Sobolev Spaces

主讲人: 李琦

liqihao@chd.edu.cn

School of Science, Chang'an University

Outline

1. SOBOLEV SPACES
   - Lebesgue spaces
   - Weak Derivative
   - Sobolev Spaces
   - Inclusion Relations and Sobolev's Inequality
   - Traces of Sobolev functions
   - Poincare's and Friedrichs' inequalities
   - Negative norms and duality

Function Spaces

As we have seen, the regularity of the solution of partial differential equations plays a crucial role in how well it can be approximated numerically. This regularity can be described by the two properties of (Lebesgue-) integrability and differentiability.

Definition (Lebesgue spaces)

Let Ω be an open subset of $\mathbb{R}^n$, $n \in \mathbb{N}$. We recall that for $1 \leq n \leq \infty$,

$$L^p(\Omega) := \left\{ f \text{ measurable } : \|f\|_{L^p(\Omega)} < \infty \right\}$$

with

$$\|f\|_{L^p(\Omega)} := \left(\int_\Omega |f(x)|^p dx \right)^{\frac{1}{p}} \qquad \text{for } 1 \leq p < \infty,$$

$$\|f\|_{L^\infty(\Omega)} := \operatorname*{ess\,sup}_{x \in \Omega} |f(x)|,$$

are Banach spaces of (equivalence classes up to equality apart from a set of zero measure of) Lebesgue-integrable functions.

Famous and useful inequalities

Minkowski's Inequality

For $1 \le p \le \infty$ and $f, g \in L^p(\Omega)$, we have

$$\|f + g\|_{L^p(\Omega)} \le \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}.$$

Hölder's Inequality

For $1 \le p, q \le \infty$ such that $1 = 1/p + 1/q$, (with $\infty^{-1} := 0$) if $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, then $fg \in L^1(\Omega)$ and

$$\|fg\|_{L^1(\Omega)} \le \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}.$$

For bounded Ω, this implies that $L^p(\Omega) \hookrightarrow L^q(\Omega)$ for $p \ge q$.

Schwarz's Inequality

This is simply Hölder's inequality in the special case $p = q = 2$, viz. if $f, g \in L^2(\Omega)$ then $fg \in L^1(\Omega)$ and

$$\int_\Omega |f(x)g(x)|\, dx \le \|f\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)}.$$

We now consider functions which are continuously differentiable. It will be convenient to use a **multi-index**

$$\alpha := (\alpha_1, \cdots, \alpha_n) \in \mathbb{N},$$

for which we define its **length** $|\alpha| := \sum_{i=0}^{n} \alpha_i$, to describe the (partial) **derivative of order** $|\alpha|$,

$$D^\alpha f(x_1, \ldots, x_n) := \frac{\partial^{|\alpha|} f(x_1, \ldots, x_n)}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

For brevity, we will often write $\partial_i := \frac{\partial}{\partial x_i}$. We denote by $C^k(\Omega)$ the set of all continuous functions f for which $D^\alpha f$ is continuous for all $|\alpha| \leq k$.

If Ω is bounded, $C^k(\overline{\Omega})$ is the set of all functions in $C^k(\Omega)$ for which all $D^\alpha f$ can be extended to a continous function on $\overline{\Omega}$, the closure of Ω. These spaces are Banach spaces if equipped with the norm

$$\|f\|_{C^k(\bar{\Omega})} = \sum_{|\alpha| \leq k} \sup_{x \in \bar{\Omega}} |D^\alpha f(x)|.$$

Support

Let us introduce the concept of the **support** of a function defined on some domain in $\mathbb{R}^n$. For a continuous function, u, this is the closure of the (open) set $\{x : u(x) \neq 0\}$. If this is a compact set (i.e., if it is bounded) and it is a subset of the *interior* of a set, Ω, then u is said to have "**compact support**" with respect to Ω.

Define $C_0^k(\overline{\Omega})$ as the space of all $f \in C^k(\overline{\Omega})$ whose support (the closure of $\{x \in \Omega : f(x) \neq 0\}$) is a compact subset of Ω, as well as

$$C_0^\infty(\overline{\Omega}) = \bigcap_{k \geq 0} C_0^k(\overline{\Omega})$$

(and similarly $C^\infty(\overline{\Omega})$) .

We will also use the space

$$L^1_{loc}(\Omega) := \left\{ f : f|_K \in L^1(K) \text{ for all compact } K \subset \Omega \right\}.$$

Weak derivative

If we are interested in weak solutions, it is clear that the Hölder spaces entail a too strong notion of (pointwise) differentiability. All we required is that the derivative is integrable, and that an integration by parts is meaningful. This motivates the following definition: A function $f \in L^1_{loc}(\Omega)$ has a **weak derivative** if there exists $g \in L^1_{loc}(\Omega)$ such that

$$\int_\Omega g(x)\varphi(x)dx = (-1)^{|\alpha|}\int_\Omega f(x)D^\alpha\varphi(x)dx \tag{1.1}$$

for all $\varphi \in C_0^\infty(\overline{\Omega})$.

In this case, the **weak derivative is (uniquely) defined as** $D^\alpha f := g$. For $f \in C^k(\Omega)$, the weak derivative coincides with the usual (pointwise) derivative (justifying the abuse of notation), but the weak derivative exists for a larger class of functions such as continuous and piecewise smooth functions.

Example

Take $\Omega = [-1, 1]$, and $f(x) = 1 - |x|$. We claim that Df exists and is given by

$$g(x) = \begin{cases} 1 & x < 0, \\ -1 & x > 0. \end{cases}$$

To see this, we break the interval $[-1, 1]$ into the two parts in which f is smooth, and we integrate by parts. Let $\phi \in C_0^\infty(\Omega)$, then

$$\begin{aligned}
\int_{-1}^{1} f(x)\phi'(x)dx &= \int_{-1}^{0} f(x)\phi'(x)dx + \int_{0}^{1} f(x)\phi'(x)dx \\
&= -\int_{-1}^{0} (+1)\phi(x)dx + f\phi\big|_{-1}^{0} - \int_{0}^{1} (-1)\phi(x)dx + f\phi\big|_{0}^{1} \\
&= -\int_{-1}^{1} g(x)\phi(x)dx + f\phi(-0) - f\phi(+0) \\
&= -\int_{-1}^{1} g(x)\phi(x)dx,
\end{aligned}$$

because f is continuous at 0 .

Relationship of classic derivatives and weak derivative

Lemma

If $u \in C^{|\alpha|}(\Omega)$, then its weak derivative $D^\alpha u$ exists and is given by the corresponding classical derivative.

One can see that roughly speaking, the new definition of the derivative is the same as the old one wherever the function being differentiated is regular enough.

Sobolev spaces

Using the notion of weak derivative, we define the Sobolev spaces.

Definition

Let k be a non-negative integer, and let $f \in L^1_{loc}(\Omega)$. Suppose that the weak derivatives $D^\alpha_w f$ exist for all $|\alpha| \leq k$. Define the **Sobolev norm**

$$\|f\|_{W^k_p(\Omega)} := \left(\sum_{|\alpha| \leq k} \|D^\alpha_w f\|^p_{L^p(\Omega)} \right)^{1/p}$$

in the case $1 \leq p < \infty$, and in the case $p = \infty$

$$\|f\|_{W^k_\infty(\Omega)} := \sum_{|\alpha| \leq k} \|D^\alpha_w f\|_{L^\infty(\Omega)}.$$

In either case, we define the **Sobolev spaces** via

$$W^k_p(\Omega) := \left\{ f \in L^1_{loc}(\Omega) : \|f\|_{W^k_p(\Omega)} < \infty \right\}.$$

We shall also use the corresponding **semi-norms**

$$|f|_{W^{k,p}(\Omega)} := \left(\sum_{|\alpha|=k} \|D_w^\alpha f\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}} \quad \text{for } 1 \leq p < \infty,$$

$$|f|_{W^{k,\infty}(\Omega)} := \sum_{|\alpha|=k} \|D_w^\alpha f\|_{L^\infty(\Omega)} .$$

$W_p^k(\Omega)$ can be written as $W^{k,p}(\Omega)$ as well. When $p = 2$, $W^{k,2}(\Omega)$ is written as $H^k(\Omega)$. Usually, $\|f\|_{k,2,\Omega}$ and $|f|_{k,2,\Omega}$ are replaced by $\|f\|_{k,\Omega}$ and $|f|_{k,\Omega}$, or even $\|f\|_k$ and $|f|_k$, when no confusion.

Completeness of Sobolev spaces

Theorem

The Sobolev space $W^{k,p}(\Omega)$ is a Banach space for $1 \leq p \leq \infty$, while $H^m(\Omega)$ is a Hilbert space, with the inner product

$$(u, v)_m = \sum_{|\alpha| \leq m} (D^\alpha u, D^\alpha v) .$$

This theorem shows that Sobolev space is complete.

Theorem

Let Ω be any open set. Then $C^\infty(\Omega) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$ for $p < \infty$. If Ω is a domain with Lipschitz boundary, then $C^\infty(\overline{\Omega}) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$ for $p < \infty$.

This theorem indicates that the closure of $C^\infty(\Omega)$ under the norm $\|\cdot\|_{W^{k,p}(\Omega)}$ is the space $W^{k,p}(\Omega)$. Motivated by this, the closure of $C_0^\infty(\Omega)$ under the norm $\|\cdot\|_{W^{k,p}(\Omega)}$ is defined to be the space $W_0^{k,p}(\Omega)$. $W_0^{k,p}(\Omega)$ is also a Banach space. And we have

$$W_0^{k,p}(\Omega) = \left\{ v \in W^{k,p}(\Omega); D^\alpha v = 0 \ \text{ on } \partial\Omega \text{ for all } \alpha \text{ with } |\alpha| \le k-1 \right\}.$$

Inclusion Relations

The following properties will be useful:

Proposition

Suppose that Ω is any domain, k and m are nonnegative integers satisfying $k \leq m$, and p is any real number satisfying $1 \leq p \leq \infty$. Then $W^{m,p}(\Omega) \subset W^{k,p}(\Omega)$

Proposition

Suppose that Ω is a bounded domain, k is a nonnegative integer, and p and q are real numbers satisfying $1 \leq p \leq q \leq \infty$. Then $W^{k,q}(\Omega) \subset W^{k,p}(\Omega)$

However, there are more subtle relations among the Sobolev spaces. For example, there are cases when $k < m$ and $p > q$ and $W^{m,q}(\Omega) \subset W^{k,p}(\Omega)$. The existence of Sobolev derivatives implies a stronger integrability condition of a function. Before we state the result, we must introduce a **regularity condition on the domain boundary** for the result to be true.

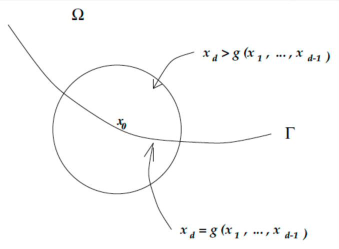
Smoothness of the boundary

Definition

Let Ω be open and bounded in $\mathbb{R}^d$, and let V denote a function space on $\mathbb{R}^{d-1}$. We say $\partial\Omega$ is of class V if for each point $x_0 \in \partial\Omega$, there exist an $r > 0$ and a function $g \in V$ such that upon a transformation of the coordinate system if necessary, we have

$$\Omega \cap B(x_0, r) = \{x \in B(x_0, r)\,; x_d > g(x_1, \cdots, x_{d-1})\}\,.$$

Here, $B(x_0, r)$ denotes the d-dimensional ball centered at x_0 with radius r. In particular, when V consists of Lipschitz continuous functions, we say Ω is a **Lipschitz domain**. When V consists of C^k functions, we say Ω is a C^k domain.



Another description

Definition

A function $f : \Omega \to \mathbb{R}^m$ is **Lipschitz continuous** in Ω if there is a constant $C > 0$ such that

$$\|f(x) - f(y)\| \leq C\|x - y\| \quad \forall x, y \in \Omega \subset \mathbb{R}^d$$

where d and m are two positive integers.

Definition

A hypersurface in $\mathbb{R}^d$ is a graph if it can be represented by a function g in the form

$$x_i = g\left(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d\right), \left(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d\right) \in D$$

for some $i (1 \leq i \leq d)$ and domain $D \subset \mathbb{R}^{d-1}$.

Definition

A domain $\Omega \subset \mathbb{R}^d$ is termed a **Lipschitz domain** if for each x in the boundary Γ of Ω, there is an open subset $O_x \subset \mathbb{R}^d$ containing x such that $O_x \cap \Gamma$ can be represented by the graph of a Lipschitz continuous function.

Examples

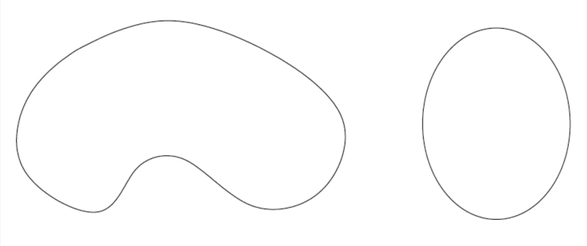


Figure: Smooth domains

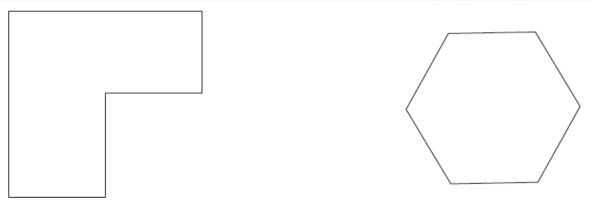


Figure: Lipschitz domains

Examples

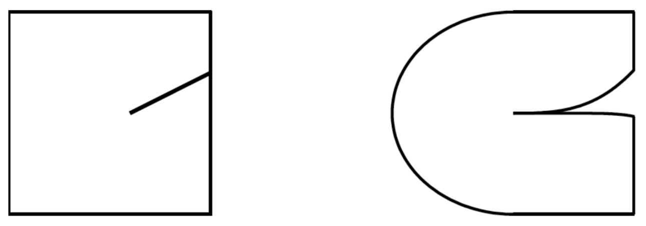


Figure: Non-Lipschitz/crack domains

Extension theorem

Theorem

Suppose that Ω *has a Lipschitz boundary. Then there is an extension mapping* $E : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^n)$ *defined for all non-negative integers* k *and real numbers* p *in the range* $1 \leq p \leq \infty$ *satisfying* $Ev|_\Omega = v$ *for all* $v \in W^{k,p}(\Omega)$ *and*

$$\|Ev\|_{W^{k,p}(\mathbb{R}^n)} \leq C\|v\|_{W^{k,p}(\Omega)}$$

where C *is independent of* v

Sobolev's inequality

Theorem (Sobolev's inequality)

Let Ω be an n-dimensional domain with Lipschitz boundary, let k be a positive integer and let p be a real number in the range $1 \leq p < \infty$ such that

$$\begin{aligned} k \geq n \quad &\textit{when} \quad p = 1 \\ k > n/p \quad &\textit{when} \quad p > 1 \end{aligned}$$

Then there is a constant C such that for all $u \in W^{k,p}(\Omega)$

$$\|u\|_{L^\infty(\Omega)} \leq C\|u\|_{W^{k,p}(\Omega)}.$$

Moreover, there is a continuous function in the $L^\infty(\Omega)$ equivalence class of u.

This result says that any function with suitably regular weak derivatives may be viewed as a continuous, bounded function. When $n = 1$, Sobolev' s inequality says that the integrability of first-order derivatives to any power $p \geq 1$ is sufficient to guarantee continuity.

Corollary

Let Ω be an n-dimensional domain with Lipschitz boundary, and let k and m be positive integers satisfying $m < k$ and let p be a real number in the range $1 \leq p < \infty$ such that

$$\begin{aligned} k - m \geq n \quad & \textit{when} \quad p = 1 \\ k - m > n/p \quad & \textit{when} \quad p > 1 \end{aligned}$$

Then there is a constant C such that for all $u \in W^{k,p}(\Omega)$

$$\|u\|_{W^{m,\infty}(\Omega)} \leq C\|u\|_{W^{k,p}(\Omega)}.$$

Moreover, there is a C^m function in the $L^p(\Omega)$ equivalence class of u.

Remark

If $\partial\Omega$ is not Lipschitz continuous, then the Sobolev's Inequality does not hold.

Embedding between function spaces

Definition

Let V and W be two Banach spaces with $V \subset W$. We say the space V is continuously embedded in W and write $V \hookrightarrow W$, if

$$\|v\|_W \leq C\|v\|_V \quad \forall v \in V.$$

We say the space V is compactly embedded in W and write $V \hookrightarrow\hookrightarrow W$, if above inequality holds and each bounded sequence in V has a convergent subsequence in W.

If $V \hookrightarrow W$, the functions in V are more smooth than the remaining functions in W. A simple example is

$$H^1(\Omega) \hookrightarrow L^2(\Omega) \text{ and } H^1(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega).$$

The following theorem is on embedding and compact embedding of Sobolev spaces.

Sobolev's embedding theorems

Theorem

Let m be positive integers, $1 \le p, q < \infty$, and $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Then the following embeddings are continuous:

$$W^{m,p}(\Omega) \hookrightarrow \begin{cases} L^q(\Omega) & \text{if } mp < n \text{ and } p \le q \le \frac{np}{n-p}, \\ L^q(\Omega) & \text{if } mp = n \text{ and } p \le q < \infty, \\ C^0(\bar{\Omega}) & \text{if } mp > n. \end{cases}$$

Moreover, the following embeddings are compact:

$$W^{m,p}(\Omega) \hookrightarrow\hookrightarrow \begin{cases} L^q(\Omega) & \text{if } mp \le n \text{ and } 1 \le q < \frac{np}{n-mp}, \\ L^q(\Omega) & \text{if } mp = n \text{ and } 1 \le q < \infty, \\ C^0(\bar{\Omega}) & \text{if } mp > n. \end{cases}$$

In particular, the embedding $W^{k,p}(\Omega) \hookrightarrow W^{k-1,p}(\Omega)$ is continuous and compact.

For $n = 2$, $\quad H^2(\Omega) \hookrightarrow C(\bar{\Omega})$, $\quad H^2(\Omega) \hookrightarrow\hookrightarrow C(\bar{\Omega})$, $\quad$ but $H^1(\Omega) \hookrightarrow C(\bar{\Omega})$.

Trace thoerem

Sobolev spaces are defined through $L^p(\Omega)$ spaces. Hence Sobolev functions are uniquely defined only a.e. in Ω. Note that the boundary $\partial\Omega$ has measure zero in $\mathbb{R}^n$, it seems the boundary value of a Sobolev function is not well-defined. Nevertheless, it is possible to define the trace of a Sobolev function on the boundary in such a way that for a Sobolev function that is continuous up to the boundary, its trace coincides with its boundary-value.

Theorem (trace theorem)

Suppose that Ω has a Lipschitz boundary, and that p is a real number in the range $1 \le p \le \infty$. Then there is a constant, C, such that

$$\|v\|_{L^p(\partial\Omega)} \le C\|v\|_{L^p(\Omega)}^{1-1/p}\|v\|_{W^{1,p}(\Omega)}^{1/p} \quad \forall v \in W^{1,p}(\Omega).$$

Trace thoerem

Theorem

For $m \geq 1$ *and a smooth domain* Ω *then the following embedding are continuous: for* $mp < n$*,*

$$W^{m,p}(\Omega) \hookrightarrow W^{m-\frac{1}{p},p}(\partial\Omega) \hookrightarrow L^q(\partial\Omega) \quad \textit{for} \quad 1 \leq q \leq \frac{(n-1)p}{n-mp}$$

and there exists some constant C *such that*

$$\|v\|_{L^q(\partial\Omega)} \leq C\|v\|_{W^{m,p}(\Omega)},$$

while for $mp = n$*,*

$$W^{m,p}(\Omega) \hookrightarrow W^{m-\frac{1}{p},p}(\partial\Omega) \hookrightarrow L^q(\partial\Omega) \quad \textit{for} \quad 1 \leq q < \infty$$

and there exists some constant C *such that*

$$\|v\|_{L^q(\partial\Omega)} \leq C\|v\|_{W^{m,p}(\Omega)}.$$

We will use the notation $\mathring{W}^{1,p}(\Omega)$ to denote the subset of $W_p^1(\Omega)$, consisting of functions whose trace on $\partial\Omega$ is zero, that is

$$\mathring{W}^{1,p}(\Omega) = \left\{ v \in W^{1,p}(\Omega) : v\big|_{\partial\Omega} = 0 \text{ in } L^2(\partial\Omega) \right\}.$$

Similarly, we let $\mathring{W}^{k,p}(\Omega)$ denote the subset of $W^{k,p}(\Omega)$ consisting of functions whose derivatives of order $k-1$ are in $\mathring{W}^{1,p}(\Omega)$, i.e.

$$\mathring{W}^{k,p}(\Omega) = \left\{ v \in W^{k,p}(\Omega) : \, v^{(\alpha)}\Big|_{\partial\Omega} = 0 \text{ in } L^2(\partial\Omega) \quad \forall |\alpha| < k \right\}.$$

Poincare's and Friedrichs' inequalities

For any $u \in W_0^{1,p}(\Omega), 1 \le p < \infty$, we have

$$\|u\|_{L^p(\Omega)} \le C(\Omega)|u|_{W^{1,p}};$$

while for any $u \in W^{1,p}(\Omega), 1 \le p < \infty$,

$$\|u - u_\Omega\|_{L^p(\Omega)} \le C(\Omega)|u|_{W^{1,p}};$$

where u_Ω is the average of u over Ω, namely

$$u_\Omega = \frac{1}{|\Omega|}\int_\Omega u(x)dx.$$

Therefore, the Poincare-Friedrichs inequality holds:

$$\|u\|_{1,\Omega} \le C|u|_{1,\Omega}, \quad \forall u \in H_0^1(\Omega).$$

From this inequality it follows that the seminorm $|\cdot|_{1,\Omega}$ is a norm on $H_0^1(\Omega)$, equivalent to the usual $H^1(\Omega)$-norm.

Norm for the dual space

We introduce ideas that lead to the definition of Sobolev spaces $W^{k,p}$ for negative integers k.

The **dual space**, B', to a Banach space, B, is a set of all continuous linear functionals on B. The following observation simplifies the characterization of such functionals.

Proposition

A linear functional, L, on a Banach space, B, ***is continuous if and only if it is bounded****, i.e., if there is a finite constant C such that*

$$|L(v)| \le C\|v\|_B \quad \forall v \in B.$$

For a continuous linear functional, L, on a Banach space, B, the proposition states that the following quantity is always finite:

$$\|L\|_{B'} := \sup_{0 \neq v \in B} \frac{L(v)}{\|v\|_B}.$$

Example

One of the key results of Lebesgue integration theory is that the dual spaces of L^p can be easily identified, for $1 \leq p < \infty$. From Hölder's inequality, any function $f \in L^q(\Omega)$ (where $\frac{1}{q} + \frac{1}{p} = 1$ defines the **dual index**, q, to p) can be viewed as a continuous linear functional via

$$L(v) = \int_\Omega v(x) f(x) dx, \quad v \in L^p(\Omega).$$

One version of the Riesz Representation Theorem states that all continuous linear functionals on $L^p(\Omega)$ arise in this way, i.e., so $(L^p(\Omega))'$ can be viewed as the dual space of $L^p(\Omega)$, that $(L^p(\Omega))'$ is isomorphic to $L^q(\Omega)$.

Definition

For $1 \leq q \leq \infty$ and a positive integer r, the dual space of the Sobolev space $W^{r,q}(\Omega)$ is indicated by $W^{-r,q'}(\Omega)$, where q' is the dual index of q. The norm on $W^{-r,q'}(\Omega)$ is defined via **duality**:

$$\|L\|_{W^{-r,q'}(\Omega)} = \sup_{0 \neq v \in W^{r,q}(\Omega)} \frac{L(v)}{\|v\|_{W^{r,q}(\Omega)}}, \quad L \in W^{-r,q'}(\Omega).$$